MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY,

CONTAINING

PAPERS, ABSTRACTS OF PAPERS, AND

REPORTS OF THE PROCEEDINGS

OF THE SOCIETY

FROM NOVEMBER 1917 *TO NOVEMBER* 1918.

VOL. LXXVIII.

PUBLISHED BY THE ROYAL ASTRONOMICAL SOCIETY,
BURLINGTON HOUSE, LONDON, W. 1,
AND SOLD BY
WILLIAM WESLEY & SON, 28 ESSEX STREET,
STRAND, LONDON, W.C. 2.
1918.

PRINTED IN GREAT BRITAIN BY NEILL AND CO., LTD., EDINBURGH.

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OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXVIII. NOVEMBER 9, 1917. No. 1

Major P. A. MACMAHON, D.Sc., Sc.D., LL.D., F.R.S., PRESIDENT, in the Chair.

Philip Francis Everitt, B.Sc., 29 Leaside Avenue, Muswell Hill, N.W. 10;

E. H. S. Flood, Canadian Trade Commissioner, Barbados, West Indies; and

Lieut. Hugh Duncan Grant, R.N.V.R., L.A., F.R.G.S., F.R.Met.Soc., Admiralty Observatory, Naval Meteorological Service, Admiralty, Whitehall, S.W. 1,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

The Rev. Arthur Harold Young Baxter, M.A.Cantab., Tresillion, Mylor, near Falmouth, Cornwall (proposed by Lieut.-Col. F. J. M. Stratton);

Walter George Bell, Journalist (*The Daily Telegraph*), 31 Baskerville Road, S.W. 18 (proposed by H. P. Hollis);

Capt. Charles John Philip Cave, R.E., F.R.Met.Soc., Ditcham Park, Petersfield (proposed by D. Brunt);

George Craig, F.I.C., 95 Bath Street, Glasgow (proposed by T. C. Hudson);

John Crighton, B.Sc., War Signal Station, Beachy Head, (proposed by Professor R. A. Sampson);

Henry Mason Garner, B.A., Isaac Newton Student of the University of Cambridge, 88 Lee Road, Blackheath, S.E. 3 (proposed by Arthur Berry);

Alfred Holness, Publisher, 13 and 14 Paternoster Row, E.C. 4 (proposed by E. Walter Maunder);

Alfred George Maddock, Sub.-Lieut., R.N.V.R., F.R.Met.Soc., Meteorological Office, Royal Naval Air Service, 6 Brooklyn Terrace, Keyham, Devonport, and R.N. Air Station, Isle of Grain, Kent (proposed by J. E. Evans);

William George de Winton Mitchell, B.Sc., The Grammar School, Newbury, Berkshire, and 52 Craven Road, Newbury (proposed by W. J. Greenstreet); and

The Rev. Roger Conyers Morrell, M.A., L.Th., F.R.G.S., F.Z.S., St. Thomas's Vicarage, Camden Town, N.W. (proposed by the Rev. G. H. Lancaster).

One hundred and fifteen presents were announced as having been received since the last meeting, including, amongst others:—

Walter Goodacre, Map of the Moon, presented by Mr. Goodacre; T. E. Heath, The distances, absolute magnitudes, and Spectra of 734 Stars, arranged for use with ordinary Star maps, presented by the author; Ulugh Beg's Catalogue of Stars, revised from all Persian manuscripts existing in Great Britain, with a vocabulary of Persian and Arabic words, by E. B. Knobel, presented by Mr. Knobel.

Photographs of the Spectroheliograph used at Srinagar, and photographs of the great prominence of 1916 May 26, taken by Mr. Evershed and Dr. Royds (16 lantern slides), presented by the Kodaikánal Observatory; Photographs of the Spectra of the Moon and Jupiter, by Professor L. Becker, presented by the Glasgow Observatory.

On Einstein's Theory of Gravitation, and its Astronomical Consequences. Third Paper.* By W. de Sitter, Assoc. R.A.S.

Contents of Third Paper.

1. On the relativity of inertia. New form of the field-equations. Two solutions A and B of these equations.
2. On space with constant positive curvature. Comparison of the two systems A and B.
3. Rays of light and parallax in the two systems. Hyperbolical space.
4. Motion of a material particle in the inertial field of the two systems. Further comparison of the two systems.
5. Differential equations for the gravitational field of the sun. Approximate integration of these equations.
6. Estimates of R in the system A.
7. Estimates of R in the system B.

1. In Einstein's theory of general relativity there is no essential difference between gravitation and inertia. The combined effect of the two is described by the fundamental tensor $g_{\mu\nu}$, and how much of it is to be called inertia and how much gravitation is entirely arbitrary. We might abolish one of the two words, and call the whole by one name only. Nevertheless it is convenient to continue to make a difference. Part of the $g_{\mu\nu}$ can be directly traced to the effect of known material bodies, and the common usage is to call this part "gravitation," and the rest "inertia." Then, if we take as a system of reference three rectangular cartesian space co-ordinates and the time multiplied by c (the velocity of light *in vacuo*), we know that, in that portion of the four-dimensional time-space which is accessible to our observations, the $g_{\mu\nu}$ of pure inertia are, within certain limits of uncertainty,

$$(1) \qquad \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{vmatrix}$$

In our immediate neighbourhood, within the solar system, the limits of uncertainty are very narrow: say the eighth decimal

* See first paper, *M.N.*, vol. lxxvi. p. 699; second paper, *M.N.*, vol. lxxvii. p. 155. The present paper gives an account of the questions treated in the following communications:—

A. Einstein, "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie," *Sitzungsber. Berlin*, 1917 Feb. 8, p. 142.

W. de Sitter, "On the Relativity of Inertia, remarks concerning Einstein's latest Hypothesis," *Proc. Akad. Amsterdam*, 1917 March 31, vol. xix. p. 1217.

W. de Sitter, "On the Curvature of Space," *Proc. Akad. Amsterdam*, 1917 June 30, vol. xx. (not yet published in English).

The notations used are the same as in the first and second papers. We may recall that $\delta_{\mu\mu}=1$, $\delta_{\mu\nu}=0$ for $\mu\neq\nu$, and that Σ is a sum from 1 to 4, and Σ' from 1 to 3.

place. As we get further away in space, or in time, or in both, the limits become wider: at a distance of a million light-years we can perhaps only guarantee the second decimal place.* How the $g_{\mu\nu}$ are in those portions of space and time to which our observations have not yet penetrated, we do not know, and how they are at infinity (of space or of time) we shall never know. All assumptions regarding the values of the $g_{\mu\nu}$ at infinity are therefore extrapolations, which we are free to choose in accordance with theoretical or philosophical requirements.

The extrapolation which most naturally offers itself, and which is also tacitly made in Newton's theory of inertia, is that the $g_{\mu\nu}$ retain the values (1) for all distances and times up to infinity. It has been pointed out in the second paper† that in this theory inertia is not relative. The values (1) are not invariant: the boundary-values of the $g_{\mu\nu}$ at infinity are different in different systems of co-ordinates. Einstein and others have therefore tried to find another extrapolation, by which the $g_{\mu\nu}$, while in our neighbourhood retaining the values (1) with the approximation demanded by the observations, would at infinity degenerate to a set of values which would be the same for all systems of reference.

The $g_{\mu\nu}$ are determined by the field-equations, which in Einstein's theory of 1915 are

$$G_{\mu\nu} = -\kappa T_{\mu\nu} + \tfrac{1}{2}\kappa g_{\mu\nu} T, \tag{2}$$

or

$$G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G = -\kappa T_{\mu\nu}, \tag{2'}$$

and

$$G = \kappa T.$$

Once the system of reference of space- and time-variables has been chosen, these equations determine the $g_{\mu\nu}$ apart from constants of integration, or boundary-conditions at infinity. Only the deviations of the actual $g_{\mu\nu}$ from these values at infinity are thus due to the effect of matter, through the mechanism of the equations (2) or (2'). If at infinity all $g_{\mu\nu}$ were *zero*, then we could truly say that the *whole* of inertia, as well as gravitation, is thus produced. This is

* There are two criteria by which we can judge the value of the fundamental tensor at great distances from us. The frequency of light-vibrations is proportional to $\sqrt{g_{44}}$. Consequently, objects in whose spectra we are able to identify definite spectral lines must be situated in a portion of space where g_{44} is still of the order of unity. The motion of material particles, on the other hand, depends on all $g_{\mu\nu}$. We know that the relative velocities of the fixed stars are small. From this we conclude that also the accelerations are small. Let the velocities be of the order α, and let g_{44} be of the order γ, and $g_{ij}+\delta_{ij}$ of the order $\beta(i, j = 1, 2, 3)$. Then the accelerations contain terms of the order γ, γ^2, $\beta \cdot \gamma$, $\alpha^2 \cdot \gamma$, $\alpha^2 \cdot \beta$, etc., but none of the order β. Thus here also we can only be sure of the smallness of γ, and not of β. Within the solar system the case is different, for there we have not only a statistical knowledge of the velocities, but we know the accelerations themselves; and our observations are so exact as to carry us to quantities of the second order. Consequently, we can be sure of g_{ij} to the first order, and of g_{44} to the second, the first order corresponding to about 10^{-8}.

† *M.N.*, vol. lxxvii. pp. 181–183.

the reasoning which has led to the postulate that at infinity all $g_{\mu\nu}$ shall be zero. I have called this the *mathematical postulate of relativity of inertia.*

If all matter were destroyed, with the exception of one material particle, then would this particle have inertia or not? The school of Mach requires the answer *No.* If, however, by "all matter" is meant all matter known to us, stars, nebulæ, clusters, etc., then the observations very decidedly give the answer *Yes.* The followers of Mach * are therefore compelled to assume the existence of still more matter. This matter, however, fulfils no other purpose than to enable us to suppose it not to exist, and to assert that in that case there would be no inertia. This point of view, which denies the logical possibility of the existence of a world without matter, I call the *material postulate of relativity of inertia.* The hypothetical matter introduced in accordance with it I call *world-matter.* Einstein originally supposed that the desired effect could be brought about by very large masses at very large distances. He has, however, now convinced himself that this is not possible. In the solution which he now proposes, the world-matter is not accumulated at the boundary of the universe, but distributed over the whole world, which is finite, though unlimited. Its density (in natural measure) is constant, when sufficiently large units of space are used to measure it. Locally its distribution may be very unhomogeneous. In fact, there is no essential difference between the nature of ordinary gravitating matter and the world-matter. Ordinary matter, the sun, stars, etc., are only condensed world-matter, and it is possible, though not necessary, to assume all world-matter to be so condensed. In this theory "inertia" is produced by the whole of the world-matter, and "gravitation" by its local deviations from homogeneity.

In Einstein's new solution the three-dimensional world is not infinite, but spherical.† Thus no boundary conditions at infinity are required. From the point of view of the theory of relativity it seems at first sight to be incorrect to say: the world *is* finite, since by a transformation of co-ordinates it can be made infinite, euclidean, or hyperbolical. Such transformations, however, leave the invariant G unaltered, and consequently also after the introduction of euclidean or hyperbolical co-ordinates the world remains finite and spherical in natural measure. The length of the semi-axis of x_1 in natural measure is

$$L_1 = \int_0^\infty \sqrt{-g_{11}}\, dx_1.$$

If this is to be finite, it is necessary that g_{11} shall become zero for $x_1 = \infty$; and inversely, if g_{11} becomes zero of a sufficiently high order for $x_1 = \infty$, then L_1 is finite. It is thus evident that the

* Mach himself still thought that the fixed stars would be sufficient. This, however, is not so.

† Or elliptical, see below, art. 2.

condition that the $g_{\mu\nu}$ shall be zero at infinity is equivalent to the finiteness of the world in natural measure.

It is found, however, that the $g_{\mu\nu}$ of this finite world do not satisfy the equations (2). Einstein is thus compelled to add a new term to these equations, which then become

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa T_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} T, \tag{3}$$

or

$$G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}(G - 2\lambda) = -\kappa T_{\mu\nu}\,; \tag{3'}$$

from which we find easily

$$G - 4\lambda = \kappa T. \tag{4}$$

If we put

$$G_{\mu\nu}' = G_{\mu\nu} - \lambda g_{\mu\nu},$$

we have

$$G' = G - 4\lambda.$$

Therefore the equations (3) and (3′) are found if in (2) or (2′) we replace $G_{\mu\nu}$ and G by $G_{\mu\nu}'$ and G′. Consequently the equations (3) can be derived from the generalised principle of Hamilton,* if we now take

$$H_3 = \int \sqrt{-g}(G - 4\lambda) d\tau.$$

All the conservative properties which follow from the principle of Hamilton thus remain true after the introduction of λ.

The curvature of the four-dimensional time-space is proportional to G. In the new theory we have $G = \kappa T + 4\lambda$: thus if there were no matter ($T = 0$), this curvature would not be zero.

Einstein's solution of the equations (3) implies the existence of a "world-matter" which fills the whole universe, as has already been mentioned. It is, however, also possible to satisfy the equations without this hypothetical world-matter. Then, of course, the "material postulate of relativity of inertia" is not satisfied, but the "mathematical postulate," which makes no mention of matter, but only requires the $g_{\mu\nu}$ to be zero at infinity, is satisfied. This is brought about by the introduction of the term with λ, and not by the world-matter, which, from this point of view, is not essential.

If we neglect all pressures and other internal forces, and if we suppose all matter to be at rest, then the tensor $T_{\mu\nu}$ becomes

$$T_{44} = g_{44}\rho, \qquad \text{all other } T_{\mu\nu} = 0, \tag{5}$$

ρ being the density in natural measure. We can put

$$\rho = \rho_0 + \rho_1, \tag{6}$$

where ρ_0 is the average density of the world-matter. If ρ_0 is positive, then ρ_1 may be positive or negative; but in the latter case the numerical value must not exceed ρ_0.

* See first paper, *M.N.*, lxxvi. p. 707.

If we wish to neglect gravitation, we must neglect ρ_1, and take ρ_0 constant. The equations (3) then become *

$$(7) \qquad \begin{cases} G_{ij} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{ij} = 0. \\ G_{44} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{44} = -\kappa\rho_0 g_{44}. \end{cases}$$

These can be satisfied by the $g_{\mu\nu}$ implied by the line-element

$$(8A) \qquad ds^2 = -dr^2 - R^2 \sin^2\frac{r}{R}\left[d\psi^2 + \sin^2\psi d\theta^2\right] + c^2dt^2,$$

if

$$(9A) \qquad \kappa\rho_0 = 2\lambda, \qquad \lambda = \frac{1}{R^2}.$$

This is Einstein's new solution.

The equations are also satisfied by

$$(8B) \qquad ds^2 = -dr^2 - R^2 \sin^2\frac{r}{R}\left[d\psi^2 + \sin^2\psi d\theta^2\right] + \cos^2\frac{r}{R}c^2dt^2,$$

if

$$(9B) \qquad \rho_0 = 0, \qquad \lambda = \frac{3}{R^2};$$

and, of course, also by

$$(8C) \qquad ds^2 = -dr^2 - r^2\left[d\psi^2 + \sin^2\psi d\theta^2\right] + c^2dt^2,$$

with

$$(9C) \qquad \rho_0 = 0, \qquad \lambda = 0.$$

This last solution (C) gives the $g_{\mu\nu}$ of the old theory of relativity, or of Newton's theory of inertia. In it three-dimensional space is euclidean, in (A) and (B) it has a constant positive curvature. In (A) there is a world-matter; in (B) and (C) we have $\rho_0 = 0$: the hypothetical world-matter does not exist.

2. If in (8A) and (8B) we put

$$(10) \qquad r = R\chi,$$

the three-dimensional line-element becomes

$$(11) \qquad d\sigma^2 = R^2\{d\chi^2 + \sin^2\chi\left[d\psi^2 + \sin^2\psi d\theta^2\right]\}.$$

This is the line-element of a three-dimensional space with a constant positive curvature, which is

$$\epsilon = \frac{1}{R^2}.$$

There are two possible forms of space with constant positive curvature, viz. the spherical space, or space of Riemann,† and the elliptical space, which has been investigated by Newcomb.‡ In the spherical space all straight lines starting from a point intersect

* The equations will be further developed in art. 5, below.

† "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," *Werke*, p. 272.

‡ "Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature in the fourth dimension," *Crelles Journal*, vol. lxxxiii. p. 293.

again in the "antipodal" point, whose distance from the first point measured along any of these lines is πR. In the elliptical space any two straight lines cannot have more than one point in common. In both forms of space the straight line is closed: its total length is $2\pi R$ in the spherical space, and πR in the elliptical space. In the spherical space the largest possible distance between two points is πR, and there is only one point, the "antipodal point," at that distance from a given point. In the elliptical space the largest possible distance is $\frac{1}{2}\pi R$, and all points at that distance from a given point lie on a straight line—the "polar line" of the point. Both spaces are finite. The total volume of the spherical space is $2\pi^2 R^3$, and of the elliptical $\pi^2 R^3$.

Einstein only mentions the spherical space, which by the two-dimensional analogy of the sphere is easier to represent to our imagination. The elliptical space is, however, really the simpler case, and it is preferable to adopt this for the physical world.* Also the spherical space would give rise to difficulties, which will be pointed out below.

We can, instead of the co-ordinates r, ψ, θ, introduce other co-ordinates by which the elliptical, or spherical, space is projected on an euclidean or on a hyperbolical space. By the transformation

$$\mathfrak{r} = R \tan \chi \tag{12}$$

the whole of the elliptical space is projected on the whole of the euclidean space.† The projection of the spherical space fills the

* This is also the opinion of Einstein (communicated to the writer by letter).

† By the transformation

$$r_1 = R \sin \chi$$

the elliptical space is made to correspond with the inside of the sphere $r_1 \leqq R$ in the euclidean space. The representation of the spherical space fills this sphere twice. If we put

$$x_1 = r_1 \sin\psi \sin\theta,$$
$$y_1 = r_1 \sin\psi \cos\theta,$$
$$z_1 = r_1 \cos\psi,$$

the co-ordinates x_1 y_1, z_1 are those used by Einstein in his paper of 1917 Feb. In these co-ordinates the three-dimensional line-element is

$$d\sigma^2 = \sum_i{}' dx_i^2 + \sum_i{}' \sum_j{}' \frac{x_i x_j dx_i dx_j}{R^2 - r_1^2}.$$

If we add

$$u_1 = R \cos \chi,$$

then x_1, y_1, z_1, u_1 are the co-ordinates used by Weierstrasz.

Riemann used the co-ordinates found by the transformation

$$r_2 = 2R \tan \tfrac{1}{2}\chi.$$

The line-element then becomes

$$d\sigma^2 = \frac{dr_2^2 + r_2^2[d\psi^2 + \sin^2\psi d\theta^2]}{(1 + r_2^2/4R^2)^2}.$$

By this transformation (which was also used in the paper by the writer of 1917 March) the whole of the spherical space corresponds to the whole of the euclidean space. The elliptical space corresponds to the inside of the sphere $r_2 \leqq 2R$.

The transformation used in the text leads to the co-ordinates of Beltrami.

euclidean space twice, the projections of antipodal points being the same.

The four-dimensional line-element in these co-ordinates is, for the two systems,

$$(13A) \qquad ds^2 = -\frac{dr^2}{(1+\epsilon r^2)^2} - \frac{r^2[d\psi^2+\sin^2\psi d\theta^2]}{1+\epsilon r^2} + c^2dt^2,$$

$$(13B) \qquad ds^2 = -\frac{dr^2}{(1+\epsilon r^2)^2} - \frac{r^2[d\psi^2+\sin^2\psi d\theta^2]}{1+\epsilon r^2} + \frac{c^2dt^2}{1+\epsilon r^2}.$$

If now we put

$$x_1 = r\sin\psi\sin\theta$$
$$x_2 = r\sin\psi\cos\theta$$
$$x_3 = r\cos\psi$$
$$x_4 = ct$$

then the $g_{\mu\nu}$ for these co-ordinates are

$$g_{ij} = -\frac{\delta_{ij}}{1+\epsilon r^2} + \frac{\epsilon x_i x_j}{(1+\epsilon r^2)^2}, \qquad \begin{cases} \text{A.} & g_{44} = 1. \\ \text{B.} & g_{44} = \dfrac{1}{1+\epsilon r^2}. \end{cases}$$

The $g_{\mu\nu}$ for $r = 0$ have the values (1) in both systems A and B. For $r = \infty$ they degenerate to

$$(1A) \qquad \begin{cases} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{cases}$$

$$(1B) \qquad \begin{cases} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{cases}$$

The set (1A) is invariant for all transformations for which (at infinity) $t' = t$; the set (1B) is invariant for *all* transformations.* It thus appears that the system A only satisfies the mathematical postulate of relativity if the latter is applied to three-dimensional space only. In other words, if we conceive the three-dimensional space (x_1, x_2, x_3) with its world-matter as movable in an absolute space, its movements can never be detected by observations: all motions of material bodies are relative to the space (x_1, x_2, x_3) with the world-matter, not to the absolute space. The world-matter thus takes the place of the absolute space in Newton's theory, or of the "inertial system." It is nothing else but this inertial system materialised. It should be pointed out that this

* With the restriction that none of the coefficients $\frac{dx_i}{dx_j'}$ becomes infinite at infinity.

relativity of inertia is in the system A only realised by making the time practically absolute. It is true that the fundamental equations of the theory, the field-equations (3) and the equations of motion, *i.e.* the differential equations of the geodetic line, remain invariant for all transformations. But only such transformations for which at infinity $t' = t$ can be carried out without altering the values (1A). In the system B, on the other hand, there is complete invariance for all transformations involving the four variables.

The system B is the four-dimensional analogy of the three-dimensional space of the system A. If we put

$$(14) \qquad ds^2 = -R^2\{d\omega^2 + \sin^2\omega(d\zeta^2 + \sin^2\zeta[d\psi^2 + \sin^2\psi d\theta^2])\},$$

the $g_{\mu\nu}$ implied by this line-element satisfy the equations (3), with the conditions (9B). In order to avoid imaginary angles, we can put

$$\omega = i\omega', \qquad \zeta = i\zeta'.$$

Then the line-element becomes *

$$(15) \qquad ds^2 = R^2\{d\omega'^2 - \sinh^2\omega'(d\zeta'^2 + \sinh^2\zeta'[d\psi^2 + \sin^2\psi d\theta^2])\}.$$

If now we put

$$\rho = R\tanh\omega'\sinh\zeta',$$
$$\tau = R\tanh\omega'\cosh\zeta',$$

then we have

$$(16) \qquad ds^2 = \frac{-(1-\epsilon\tau^2)d\rho^2 - 2\epsilon\rho\tau d\rho d\tau + (1+\epsilon\rho^2)d\tau^2}{[1+\epsilon(\rho^2-\tau^2)]^2} - \frac{\rho^2[d\psi^2 + \sin^2\psi d\tau^2]}{1+\epsilon(\rho^2-\tau^2)}.$$

* If we take

$$\begin{aligned} &x = R\sinh\omega'\sinh\zeta', \qquad &&t = R\sinh\omega'\cosh\zeta', \\ &x = r\sin\psi\sin\theta, \\ &y = r\sin\psi\cos\theta, \qquad &&u = R\cosh\omega', \\ &z = r\cos\psi, \end{aligned}$$

we have

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - du^2,$$

and

$$(a) \qquad R^2 - x^2 - y^2 - z^2 + t^2 - u^2 = 0.$$

The latter equation represents an hyperboloid (one-bladed) in the five-dimensional space (x, y, z, t, u). The projection of a point x, y, z, t, u of this hyperboloid from the point $x=y=z=t=u=0$ on the four-dimensional space $u = R$ has the co-ordinates (ξ, η, ζ, τ), where

$$\xi = \rho\sin\psi\sin\theta,$$
$$\eta = \rho\sin\psi\cos\theta,$$
$$\zeta = \rho\cos\psi.$$

This projection is limited by the "hyperbola"

$$(b) \qquad R^2 + \xi^2 + \eta^2 + \zeta^2 - \tau^2 = 0, \quad \text{or} \quad 1 + \epsilon(\rho^2 - \tau^2) = 0,$$

which is the projection of the points at infinity on the hyperboloid (*a*). The part of $u = R$ which is outside the hyperbola (*b*) is the projection of the (two-bladed) hyperboloid which is conjugated to (*a*). It will be seen from (16) that on the limiting "hyperbola" (*b*) all $g_{\mu\nu}$ become infinite.

Finally, by the transformation

$$\mathrm{R}\sin\frac{r}{\mathrm{R}}=\frac{\rho}{\sqrt{1+\epsilon(\rho^2-\tau^2)}},\qquad \mathrm{R}\sinh\frac{ct}{\mathrm{R}}=\frac{\tau}{\sqrt{1-\epsilon\tau^2}},$$

we find the formula (8B).

In the three-dimensional space, whose line-element is (11), we can transfer the origin to a point $(\chi_1, \psi_1, \theta_1)$, and the line-element expressed in co-ordinates referred to this new origin will again have the same form (11). Exactly in the same way we can in (14) transfer the origin to a point $(\omega_1, \zeta_1, \psi_1, \theta_1)$, corresponding to $(\chi_1, \psi_1, \theta_1, ct_1)$ in (8B). The line-element in the co-ordinates referred to this new origin will again have the same form (14), and this can again be transformed to new variables $\chi', \psi', \theta', ct'$, and will then again have the form (8B). Of course ct' will generally be different from ct.

In both systems A and B it is always possible, at every point of the four-dimensional time-space, to find systems of reference in which the $g_{\mu\nu}$ depend only on one space-variable (the "radius-vector"), and not on the "time." In the system A the "time" of these systems of reference is the same always and everywhere, in B it is not. In B there is no universal time; there is no essential difference between the "time" and the other three co-ordinates. None of them has any real physical meaning. In A, on the other hand, the time is essentially different from the space-variables.

3. In order further to compare the two systems, we will consider the course of rays of light. In A, if we use the co-ordinates r, ψ, θ, ct, the velocity of light is constant, and the rays of light, which are geodetic lines in the four-dimensional time-space, are also geodetic in the three-dimensional space r, ψ, θ. On triangles formed by such lines the ordinary formulæ of spherical trigonometry are applicable. Thus, if we suppose the sun to be at rest in the origin of co-ordinates, and if the distance sun-earth be called a, then the parallax * p of a star whose distance from the sun is r, is given by

$$\tan p=\sin\frac{a}{\mathrm{R}}\cot\frac{r}{\mathrm{R}};$$

or, since the square of a/R can be neglected,

$$p=\frac{a}{\mathrm{R}}\cot\frac{r}{\mathrm{R}}. \tag{17}$$

The same result is found in the reference system r, ψ, θ, ct. By the transformation (12) all straight lines remain straight in the projection. We can, moreover, easily verify that the rays of

* The parallax is 90° − A, if A is the angle at the earth, the angle at the sun being 90°. In spherical geometry, of course, 90° − A is *not* equal to the angle at the star, as it is in euclidean geometry.

light must be straight lines in the system r, ψ, θ, ct. The velocity of light in this system is

$$v = \frac{c(1+\epsilon r^2)}{\sqrt{(1+\epsilon r^2 \sin^2 V)}},$$

where V is the angle between the radius-vector and the tangent to the ray of light. The equation of the ray of light then becomes *

$$\sin V = \frac{k}{r},$$

k being a constant. This is the equation of a straight line. The parallax is thus determined by the ordinary formulas of euclidean geometry, and we have

$$p = \frac{a}{r} = \frac{a}{R} \cot \frac{r}{R},$$

which is the same as (17).

The parallax vanishes for $r = \frac{1}{2}\pi R$, *i.e.* for the largest distance which can occur in elliptical space. If we adopted the spherical space, so that still larger distances could occur, p would become negative, and for $r = \pi R$ we would have $p = -90°$.

In the system B the rays of light are *not* geodetic lines in the three-dimensional space (r, ψ, θ), nor in (r, ψ, θ). In (r, ψ, θ) the velocity of light is $v = c \cos \chi$. If now we introduce a new variable h by the condition

$$\frac{dr}{dh} = \cos \chi,$$

of which the integral is

$$\sinh \frac{h}{R} = \tan \frac{r}{R} = \frac{r}{R}, \tag{18}$$

then the velocity of light in the radial direction will be constant. The line-element becomes †

$$ds^2 = \frac{-dh^2 - R^2 \sinh^2 \frac{h}{R}[d\psi^2 + \sin^2 \psi d\theta^2] + c^2 dt^2}{\cosh^2 \frac{h}{R}}. \tag{19B}$$

* See first paper, *M.N.*, lxxvi. p. 717.

† The transformation (18) can, of course, also be applied in the system A. Then the line-element becomes

$$ds^2 = \frac{-dh^2 - R^2 \sinh^2 \frac{h}{R}[d\psi^2 + \sin^2 \psi d\theta^2]}{\cosh^2 \frac{h}{R}} + c^2 dt^2. \tag{19A}$$

In (19A) all g^{ij} become zero for $h = \infty$, but g_{44} remains 1; in (19B) g_{44} also becomes zero.

The three-dimensional space of this system of reference is the space with constant *negative* curvature, or hyperbolical space, or space of Lobatschewski. It is evident from (18) that the whole of elliptical space corresponds to the whole of the hyperbolical space; the representation of the spherical space would fill the hyperbolical space twice.

In the system of reference h, ψ, θ, ct the velocity of light is constant [in all directions, though the transformation (18) was found from the condition that it should be constant in the radial direction], and the rays of light are straight (*i.e.* geodetic) lines in the three-dimensional hyperbolical space (h, ψ, θ). This hyperbolical space thus in the system B plays the same part as the elliptical space in the system A (and the euclidean space in the system C), so far as the propagation of light is concerned. If the motion of material particles (mechanics) also is considered, then the analogy breaks down, owing to the numerator $\cosh^2 h/\mathrm{R}$.

The light-rays being straight lines, we can for the derivation of the parallax use the formulas of trigonometry in hyperbolical geometry. We thus find

$$\tan p = \sinh \frac{a}{\mathrm{R}} \coth \frac{h}{\mathrm{R}},$$

or

$$p = \frac{a}{\mathrm{R}} \coth \frac{h}{\mathrm{R}}. \tag{20}$$

It follows that in the system B the parallax of a star can never be zero. For $h = \infty$ we have $p = \frac{a}{\mathrm{R}}$. By the transformation (18) we have

$$p = \frac{a}{\mathrm{R} \sin \chi} = \frac{a}{\mathrm{r}} \sqrt{1 + \frac{\mathrm{r}^2}{\mathrm{R}^2}}. \tag{20'}$$

Thus p reaches its minimum value $\frac{a}{\mathrm{R}}$ for $\chi = \frac{1}{2}\pi$. For larger values of χ, which can only occur in the spherical space, p would increase again, and for $r = \pi\mathrm{R}$ we would have $p = 90^\circ$. In fact, if the spherical space is projected by (18) on the hyperbolical space, the projections of antipodal points coincide: a star at the antipodal point of the sun would be projected in the sun.

It may be interesting to derive the formula (20′) from the course of rays of light in the system (r, ψ, θ). In this system the velocity of light is

$$v = c \sqrt{\frac{1 + \epsilon \mathrm{r}^2}{1 + \epsilon \mathrm{r}^2 \sin^2 \mathrm{V}}}.$$

The equation of the ray of light becomes

$$\sin \mathrm{V} = \frac{a}{\mathrm{r}(1 + \epsilon \mathrm{r}^2)}.$$

The parallax is determined by the equation *

$$\frac{dp}{dr} = -\frac{\tan V}{r},$$

from which, if we neglect a^2/r^2, we find

$$p = \frac{a}{r}\sqrt{1+\epsilon r^2},$$

which is the same as (20′).

4. The equations of motion of a material particle in the field of pure inertia are the differential equations of the geodetic line, viz.

$$(21) \qquad \frac{d^2x_i}{c^2dt^2} = -\sum_p\sum_q\left[\begin{Bmatrix} pq \\ i \end{Bmatrix} - \begin{Bmatrix} pq \\ 4 \end{Bmatrix}\dot{x}_i\right]\dot{x}_p\dot{x}_q;$$

or, if we restrict ourselves to such systems of reference in which the $g_{\mu\nu}$ do not depend on $x_4 = ct$,

$$(21') \qquad \frac{d^2\dot{x}_i}{c^2dt^2} = -\begin{Bmatrix} 44 \\ i \end{Bmatrix} - \sum_p{}'\sum_q{}'\begin{Bmatrix} pq \\ i \end{Bmatrix}\dot{x}_p\dot{x}_q + 2\sum_p{}'\begin{Bmatrix} p4 \\ 4 \end{Bmatrix}\dot{x}_p\dot{x}_p.$$

In the system C, which represents Newton's theory of inertia, if we take rectangular cartesian space-co-ordinates, the g_{ij} are given by (1), and all the brackets are zero. Consequently

$$\frac{d^2x_i}{c^2dt^2} = 0.$$

The orbit of a particle under the influence of inertia without gravitation is thus a straight line in euclidean space, and the velocity is constant.

In the system A we have for the co-ordinates r, ψ, θ, ct,

$$\begin{Bmatrix} 22 \\ 1 \end{Bmatrix} = -R\sin\chi\cos\chi, \qquad \begin{Bmatrix} 33 \\ 1 \end{Bmatrix} = -R\sin\chi\cos\chi\sin^2\psi,$$

$$\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 3 \end{Bmatrix} = \frac{1}{R}\cot\chi, \quad \begin{Bmatrix} 33 \\ 2 \end{Bmatrix} = -\sin\psi\cos\psi, \quad \begin{Bmatrix} 23 \\ 3 \end{Bmatrix} = \cot\psi.$$

* Strictly speaking, r here is the distance from the star to the earth, instead of to the sun. The square of a^2/r^2 being neglected, these two distances may be interchanged. We thus have, in the notation of the first paper,

$$p = V - x.$$

Now we have (see first paper, p. 718)

$$\frac{dx}{dV} = 1 + \frac{\tan V}{r}\frac{dr}{dV},$$

or

$$\frac{dx}{dr} = \frac{dV}{dr} + \frac{\tan V}{r},$$

from which the equation for p follows immediately.

The other brackets are zero. We find:

$$\frac{d^2r}{c^2dt^2} = \mathrm{R}\sin\chi\cos\chi\left[\left(\frac{d\psi}{cdt}\right)^2 + \sin^2\psi\left(\frac{d\theta}{cdt}\right)^2\right],$$

$$\frac{d^2\theta}{c^2dt^2} = -\frac{2}{\mathrm{R}}\cot\chi\frac{dr}{cdt}\cdot\frac{d\theta}{cdt} - 2\cot\psi\frac{d\psi}{cdt}\cdot\frac{d\theta}{cdt},$$

$$\frac{d^2\psi}{c^2dt^2} = -\frac{2}{\mathrm{R}}\cot\chi\frac{dr}{cdt}\cdot\frac{d\psi}{cdt} + \sin\psi\cos\psi\left(\frac{d\theta}{cdt}\right)^2.$$

We can take $\psi = 90°$, $\frac{d\psi}{dt} = 0$. Then we find the integrals of areas and of living force:

$$(22)\qquad \begin{cases} \mathrm{R}^2\sin^2\chi\left(\frac{d\theta}{dt}\right) = \mathrm{c} \\ \mathrm{R}^2\sin^2\chi\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 = \mathrm{k}. \end{cases}$$

Eliminating dt, we find the differential equation of the orbit

$$(23)\qquad \left(\frac{dr}{d\theta}\right)^2 + \mathrm{R}^2\sin^2\chi = \frac{\mathrm{k}}{\mathrm{c}^2}\mathrm{R}^4\sin^4\chi.$$

The integral is

$$(24)\qquad \tan\chi\cos(\theta - \theta_0) = \frac{\mathrm{c}}{\mathrm{k}\mathrm{R}^2 - \mathrm{c}}.$$

This is the equation of a straight (*i.e.* geodetic) line in the spherical or elliptical space. By the second of (22) the velocity is constant. Thus in the system A a material particle under the action of inertia alone describes a straight line in elliptical space with a constant velocity.

In the case of the system B we will use the co-ordinates r, ψ, θ, ct. Then we have:

$$\left\{\begin{matrix}11\\1\end{matrix}\right\} = -\frac{2\epsilon\mathrm{r}}{1+\epsilon\mathrm{r}^2},\quad \left\{\begin{matrix}22\\1\end{matrix}\right\} = -\mathrm{r},\quad \left\{\begin{matrix}33\\1\end{matrix}\right\} = -\mathrm{r}\sin^2\psi,\quad \left\{\begin{matrix}44\\1\end{matrix}\right\} = -\epsilon\mathrm{r},$$

$$\left\{\begin{matrix}12\\2\end{matrix}\right\} = \left\{\begin{matrix}13\\3\end{matrix}\right\} = \frac{1}{\mathrm{r}},\quad \left\{\begin{matrix}33\\2\end{matrix}\right\} = -\sin\psi\cos\psi,\quad \left\{\begin{matrix}23\\3\end{matrix}\right\} = \cot\psi,\quad \left\{\begin{matrix}14\\4\end{matrix}\right\} = -\frac{\epsilon\mathrm{r}}{1+\epsilon\mathrm{r}^2}.$$

The others are zero. We find now:

$$\frac{d^2\mathrm{r}}{c^2dt^2} = \epsilon\mathrm{r} + \mathrm{r}\left[\left(\frac{d\psi}{cdt}\right)^2 + \sin^2\psi\left(\frac{d\theta}{cdt}\right)^2\right],$$

$$\frac{d^2\theta}{c^2dt^2} = -\frac{2}{\mathrm{r}}\frac{d\mathrm{r}}{cdt}\frac{d\theta}{cdt} - 2\cot\psi\frac{d\psi}{cdt}\frac{d\theta}{cdt},$$

$$\frac{d^2\psi}{c^2dt^2} = -\frac{2}{\mathrm{r}}\frac{d\mathrm{r}}{cdt}\frac{d\psi}{cdt} + \sin\psi\cos\psi\left(\frac{d\theta}{cdt}\right)^2.$$

We can again take $\psi = 90°$, $\frac{d\psi}{dt} = 0$. The integrals of areas and living force are

$$(25)\qquad \begin{cases} r^2\frac{d\theta}{dt} = c. \\ \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \epsilon r^2 + k. \end{cases}$$

The differential equation of the orbit is

$$(26)\qquad \left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{\epsilon r^2 + k}{c^2}\,.\,r^4.$$

The integration is easily effected by putting $y = \frac{1}{2r^2}$. We find

$$(27)\qquad r^2[1 + e\cos 2(\theta - \theta_0)] = \frac{2c^2}{k},$$

where

$$e = \frac{\sqrt{(4\epsilon c^2 + k^2)}}{k}.$$

This becomes a straight line in elliptical space * only if $e = 1$ or $c = 0$, *i.e.* $\frac{d\theta}{dt} = 0$. The orbit is thus only straight if it passes through the origin.

We can complete the integration by introducing an auxiliary angle u. We find the formulas

$$(28)\qquad \begin{cases} r^2 = \tfrac{1}{2}R^2k(e\cosh 2u - 1), \\ r^2\cos 2(\theta - \theta_0) = \tfrac{1}{2}R^2k(e - \cosh 2u), \\ r^2\sin 2(\theta - \theta_0) = \tfrac{1}{2}R^2k\sqrt{e^2 - 1}\sinh 2u, \\ \tan(\theta - \theta_0) = \sqrt{\frac{e+1}{e-1}}\tanh u, \\ u = \frac{t}{R} + u_0. \end{cases}$$

We have

$$\frac{dr}{dt} = \cos^2\chi\frac{dr}{dt}.$$

* In the co-ordinates h, ψ, θ (hyperbolical space) the equation (27) becomes

$$R^2\tanh^2\frac{h}{R}[k + 2\epsilon c^2 + ke\cos 2(\theta - \theta_0)] = 2c^2,$$

which is a straight line when

$$e = 1 + \frac{2\epsilon c^2}{k}.$$

Consequently, the integrals (25) expressed in the co-ordinates r, ψ, θ of elliptical space become *

$$(25') \quad \begin{cases} \mathrm{R}^2 \tan^2 \chi \dfrac{d\theta}{dt} = \mathrm{c}. \\ \left(\dfrac{dr}{dt}\right)^2 + \mathrm{R}^2 \sin^2 \chi \left(\dfrac{d\theta}{dt}\right)^2 = \sin^2 \chi \cos^2 \chi + (\mathrm{k} + \epsilon \mathrm{c}^2) \cos^4 \chi. \end{cases}$$

In the system B, therefore, a material particle under the influence of inertia alone does *not* describe a straight line with constant velocity. The orbit can only be straight if it passes through the origin, but even then the velocity is not constant. For small values of χ the equations (25′) do, however, not differ from (22). Those parts of the orbit which come within the reach of our observations therefore are sensibly straight if we adopt a sufficiently large value of R.

The velocity becomes zero for $r = \frac{1}{2}\pi \mathrm{R}$. Thus a material particle which is on the polar line of the origin can have no velocity. It also has no energy, for the energy of a material particle is

$$m \sum_p g_{p4} \frac{dx_p}{ds},$$

which also vanishes for $r = \frac{1}{2}\pi \mathrm{R}$. Also the velocity of light is zero on the polar line.

All these results sound very strange and paradoxical. They are, of course, all due to the fact that g_{44} becomes zero for $r = \frac{1}{2}\pi \mathrm{R}$. We can say that on the polar line the four-dimensional time-space is reduced to the three-dimensional space: *there is no time*, and consequently no motion.

It may be pointed out that the time taken by light to reach the distance $\frac{1}{2}\pi \mathrm{R}$ from the origin (or from any other point) is

$$\mathrm{T} = \frac{\mathrm{R}}{c} \int_0^{\frac{1}{2}\pi} \sec \chi d\chi = \infty.$$

A fortiori the time needed by a material particle for the same journey is also infinite. This also follows from the equations (28), for the distance $r = \frac{1}{2}\pi \mathrm{R}$ corresponds to $\mathrm{r} = \infty$, and consequently to $u = \pm\infty$, or $t = \pm\infty$. A particle which has not always been on the polar line can therefore only reach it after an infinite time, *i.e.* it can never reach it at all. We can thus say that all the paradoxical phenomena (or rather negations of phenomena) which

* In the co-ordinates of hyperbolical space we have

$$\mathrm{R}^2 \sinh^2 \frac{h}{\mathrm{R}} \frac{d\theta}{dt} = \mathrm{c},$$

$$\left(\frac{dh}{dt}\right)^2 + \mathrm{R}^2 \sinh^2 \frac{h}{\mathrm{R}} \left(\frac{d\theta}{dt}\right)^2 = \tanh^2 \frac{h}{\mathrm{R}} + (\mathrm{k} + \epsilon \mathrm{c}^2) \operatorname{sech}^2 \frac{h}{\mathrm{R}}.$$

The law of areas is therefore true in hyperbolical space in the system B, as it is in elliptical space in the system A, and in euclidean space in C.

have been enumerated above can only happen after the end or before the beginning of eternity.*

Of course such things as "velocity" and "energy" are relative to the system of co-ordinates. They are not tensors, and consequently different in different systems of reference. It may well be that the system r, ψ, θ, ct is not the most simple or the most convenient to describe the phenomena. When described in other co-ordinates the same results may present themselves in a different form. But the fact remains that the extrapolation according to the hypothesis B is more different from what we are used to in our neighbourhood than that according to the hypotheses A or C.

The system A satisfies the "material postulate of relativity of inertia," but it restricts the admissible transformations to those for which at infinity $t' = t$, and thus introduces a quasi-absolute time, as has been explained in art. 2. In B and C the time is entirely relative, and completely equivalent to the other three co-ordinates. In A there is a world-matter, with which the whole world is filled, and this can be in a state of equilibrium without any internal stresses or pressures, if it is entirely homogeneous and at rest. In B there may, or may not, be matter, but if there is more than one material particle these cannot be at rest, and if the whole world were filled homogeneously with matter this could not be at rest without internal pressure or stress; for if it were, we would have the system A, with $g_{44} = 1$ for all values of the four co-ordinates. The system B satisfies the "mathematical postulate" of relativity of inertia, which does not appear to admit of a simple physical interpretation.

In the system C we have no relativity of inertia at all. It cannot be denied that the introduction of the constant λ, which distinguishes the systems A and B from C, is somewhat artificial, and detracts from the simplicity and elegance of the original theory of 1915, one of whose great charms was that it embraced so much without introducing any new empirical constant.

Postscriptum to Art. 4 (*added* 1917 *October*).

[The orbit of a material particle in the system B under the influence of inertia alone is given by the equation (27). This equation represents a *hyperbola*. If by r_0 we denote the minimum value of r, and by v_0 the velocity (dr/cdt) at this point, then we have

$$e = \frac{v_0^2 + \epsilon r_0^2}{v_0^2 - \epsilon r_0^2}, \qquad c = r_0 v_0, \qquad k = v_0^2 - \epsilon r_0^2.$$

Further, if we put

$$x = r\cos(\theta - \theta_0), \qquad y = r\sin(\theta - \theta_0),$$

* In the systems of reference in which the radius-vector is measured by r (projection on euclidean space) and h (projection on hyperbolical space) they are also relegated to infinity of space.

then the equation (27) reduces to

$$\frac{x^2}{r_0^2} - \frac{y^2}{R^2 v_0^2} = 1, \tag{27'}$$

which represents a hyperbola of which the real axis is r_0 and the imaginary axis Rv_0. For a velocity of half a mile per day this latter axis still exceeds the distance of Neptune from the sun (assuming $R = 10^{12}$), and consequently for all observable phenomena the hyperbola can be treated as a straight line.

The equations (28) may be similarly transformed to

$$x = r_0 \cosh u, \qquad y = Rv_0 \sinh u, \qquad u = \frac{t}{R} + u_0. \tag{28'}$$

For $v_0 = 1$ the velocity is equal to the velocity of light, and the orbit becomes a ray of light. The rays of light are therefore (in the system of reference r, ψ, θ, ct) hyperbolas whose imaginary axis is R. It is easily verified that this is in accordance with the result found in art. 3.]

5. We will now further develop the field-equations

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa T_{\mu\nu} + \tfrac{1}{2}\kappa g_{\mu\nu} T. \tag{3}$$

We will consider no other source of gravitation than a material sphere at the origin of co-ordinates, which we will call the sun. In the systems B and C there is then no other matter than this: inside the sun we have $\rho = \rho_1$, and outside $\rho = 0$. In the system A the average density of the world-matter must remain constant. If part of it is condensed to form the sun, then the density in the neighbourhood of the sun is decreased, so that the total mass in a sufficiently large volume surrounding the sun is not affected. The mass of the sun is, however, extremely small compared with the total mass within one unit of volume of such size as must be used to measure the average density, and we can hardly postulate the total mass within each unit of volume to be *exactly* the same. We will therefore neglect the compensation, and take

$$\begin{aligned} &\text{within the sun: } \rho = \rho_0 + \rho_1, \\ &\text{outside the sun: } \rho = \rho_0. \end{aligned}$$

Although this is not strictly in accordance with Einstein's hypothesis of constant average density, it is at all events a legitimate problem to investigate the field of gravitation and inertia for this distribution of matter.

We can take

$$ds^2 = -a dy^2 - b[d\psi^2 + \sin^2\psi d\theta^2] + fc^2 dt^2,$$

where y represents any of the variables r, h, r, etc., which may be used to measure the radius-vector. The equations become somewhat simpler if we introduce

$$l = lg a, \qquad m = lg b, \qquad n = lg f.$$

We can suppose a, b, f to be functions of y and ct only. We indicate differential quotients with respect to y by accents, and with respect to ct by dots. Then we find

$$(29)\quad \begin{cases} G_{11} = m'' + \frac{1}{2}n'' + \frac{1}{2}m'(m' - l') + \frac{1}{4}n'(n' - l') \\ \qquad\qquad - \frac{a}{f}\{\frac{1}{2}\ddot{l} + \frac{1}{4}\dot{l}(\dot{l} + 2\dot{m} - \dot{n})\}, \\ \frac{a}{b}G_{22} = -\frac{a}{b} + \frac{1}{2}m'' + \frac{1}{4}m'(n' + 2m' - l') \\ \qquad\qquad - \frac{a}{f}\{\frac{1}{2}\ddot{m} + \frac{1}{4}\dot{m}(\dot{l} + 2\dot{m} - \dot{n})\}, \\ -\frac{a}{f}G_{44} = \frac{1}{2}n'' + \frac{1}{4}n'(n' + 2m' - l') \\ \qquad\qquad - \frac{a}{f}\{\ddot{m} + \frac{1}{2}\ddot{l} + \frac{1}{2}\dot{m}(\dot{m} - \dot{n}) + \frac{1}{4}\dot{l}(\dot{l} - \dot{n})\}, \\ G_{33} = \sin^2\psi \,.\, G_{22}. \end{cases}$$

We must now introduce the tensor $T_{\mu\nu}$. If we take the values (5), replacing ρ by ρ_0, the equations (3) become

$$G_{11} = -a(\lambda + \tfrac{1}{2}\kappa\rho_0),$$

$$\frac{a}{b}G_{22} = -a(\lambda + \tfrac{1}{2}\kappa\rho_0),$$

$$-\frac{a}{f}G_{44} = -a(\lambda - \tfrac{1}{2}\kappa\rho_0),$$

which are the equations (7) already given above. It is easily verified that all the different sets of $g_{\mu\nu}$ which have been given above for the inertial field satisfy these equations, if the appropriate values are taken for λ and ρ_0.

It has been found above that we can always introduce such a system of reference that a, b, f are functions of the variable y only. We can thus omit the lower lines of the expressions (29). Doing this, and using in $T_{\mu\nu}$ again ρ instead of ρ_0, I find by a slight transformation *

$$(30)\quad \begin{cases} (a) \quad n'' + n'(m' + \frac{1}{2}n' - \frac{1}{2}l') = a\kappa\rho - 2a\lambda, \\ (b) \quad m'' + \frac{1}{2}m'(m' - n' - l') = -a\kappa\rho, \\ (c) \quad -\frac{a}{b} + \frac{1}{2}m'(n' + \frac{1}{2}m') = -a\lambda. \end{cases}$$

In the equations (3′) the $g_{\mu\nu}$ occur not only in the left-hand members, but are also involved in the $T_{\mu\nu}$. In (30) we have taken the values (5) of $T_{\mu\nu}$. These correspond to the case that all matter is at rest, and not subject to any pressure or other internal forces. The fact that the $g_{\mu\nu}$ of the inertial field of the system A satisfies these equations thus proves that by inertia alone no internal stress

* We take

$$(a) = -2\frac{a}{f}G_{44}, \quad (b) = G_{11} + \frac{a}{f}G_{44}, \quad (c) = \frac{a}{b}G_{22} - \tfrac{1}{2}(b).$$

or pressure is produced in the world-matter, if this is at rest, and if ρ_0 is constant.

If the equations (30) are correct, which they only are in case the values (5) of the $T_{\mu\nu}$ are admissible, they must be dependent on each other.* If we form the combination

$$2\frac{d(c)}{dy}+2[m'-l'].(c)-[m'+n'].(b)-m'.(a),$$

then we find

$$0=a\kappa\rho n'. \tag{31}$$

The equations (30) are therefore only correct when either $\rho=0$ (as in the systems B and C outside matter), or $f=$ const. (as in A and C in the absence of gravitation). If the gravitational effect of the sun is not neglected, we cannot use the values (5) of $T_{\mu\nu}$. If in the system A we consider the world-matter as a continuous "fluid" at rest, there must be a stress or pressure in it; if it is considered as consisting of concrete material particles, these cannot be at rest. Which way of treating it is chosen, is not essential for our purpose. We will assume the world-matter to be an incompressible fluid. Then we have †

$$T_{ii}=-g_{ii}p, \qquad T_{44}=g_{44}\rho. \tag{32}$$

If this is introduced, we find, instead of (30),

$$\begin{cases} (a) & n''+n'(m'+\frac{1}{2}n'-\frac{1}{2}l')=a\kappa(\rho+3p)-2a\lambda, \\ (b) & m''+\frac{1}{2}m'(m'-n'-l')=-a\kappa(\rho+p), \\ (c) & -\dfrac{a}{b}+\frac{1}{2}m'(n'+\frac{1}{2}m')=a\kappa p-a\lambda. \end{cases} \tag{33}$$

If p is determined in accordance with the principles of Einstein's theory, the equations (33) become dependent on each other.‡ We can therefore use the equations (33), with an arbitrary

* See first paper, art. 8 (*M.N.*, lxxvi. p. 708).

† See first paper, p. 713, where we put P=0.

‡ If for a, b, f we take the values of the inertial field of the system A, viz.

$$a=1, \qquad b=R^2\sin^2\chi, \qquad f=1,$$

then the three equations (33) are found to be dependent on each other, and to reduce to

$$\lambda=\frac{3}{R^2}-\kappa\rho_0, \qquad \kappa(p+\rho_0)=\frac{2}{R^2}. \tag{α}$$

If we take $p=0$, these give the values (9A) of ρ_0 and λ. If, however, we admit a pressure in the world-matter, we can have other values of ρ_0 and λ.

If we take the values of the inertial field of the system B, viz.

$$a=1, \qquad b=R^2\sin^2\chi, \qquad f=\cos^2\chi,$$

then the three equations are again dependent on each other, and reduce to

$$\lambda=\frac{3}{R^2}-\kappa\rho_0, \qquad p+\rho_0=0. \tag{β}$$

For $p=0$ these give the values (9B), and, unless we are prepared to admit a negative pressure, this is the only solution.

These considerations originated from a remark made by Professor Lorentz

additional condition, to determine a, b, f, and p. The same combination as was used above now gives, instead of (31),

$$a\kappa[(\rho+p)n'+2p']=0. \tag{34}$$

We require only the field outside the sun. We can thus take $\rho=\rho_0$. Integrating (34), and determining the constant of integration from the condition that for $f=1$ we must have $p=0$, we find

$$p=\rho_0\left(\frac{1}{\sqrt{f}}-1\right). \tag{35}$$

If this value of p is introduced, the equations (33) become dependent on each other. We can thus use two of them, and add an arbitrary condition. If we use the co-ordinates of elliptical space, *i.e.* if we take $y=r$, we can put

$$a=1+\alpha, \quad b=R^2\sin^2\chi(1+\beta), \quad f=1+\gamma.$$

Further, instead of (9A) we now take

$$\lambda=\epsilon(1+\zeta), \qquad \kappa\rho_0=2\lambda+\epsilon.\delta,$$

where ζ and δ are constants, which must be of the same order as α, β, γ. If we neglect quantities of the second order, we have from (35)

$$p=-\tfrac{1}{2}\rho_0\gamma.$$

The equations (33), of which we use the first and the last, then become, to the first order,

$$\begin{cases} (a) & R^2\gamma''+2R\cot\chi.\gamma'+3\gamma=\delta. \\ (c) & \beta\operatorname{cosec}^2\chi-\alpha\cot^2\chi+R\cot\chi(\beta'+\gamma')+\gamma+\zeta=0. \end{cases} \tag{36}$$

From (36, a) we find *

$$\gamma=-\mu\frac{\cos 2\chi}{\sin\chi}+\tfrac{1}{3}\delta.$$

For $\chi=\frac{1}{2}\pi$ we must have $\gamma=0$. This determines δ. We find $\delta=-3\mu$. The formula thus becomes

$$\gamma=-\mu\left(\frac{\cos 2\chi}{\sin\chi}+1\right), \tag{37}$$

which, after development in powers of $1/R$, becomes

$$\gamma=-\mu R\left(\frac{1}{r}+\frac{1}{R}+\ldots\right).$$

(in a letter to the writer). Formulas equivalent to (a) also occur in a paper by Professor T. Levi-Civita, "Realta fisica di alcuni spazi normali di Bianchi" (*Rendiconti d. Acc. dei Lincei*, 1917 May, p. 521), which only came to my acquaintance after the present paper had been sent to the press.

* The complete integral has $\cos(2\chi+\omega)$ instead of $\cos 2\chi$, ω being a constant of integration, for which we take the value zero.

It thus appears that μR is the same quantity that has been called $2\lambda_0^2$ in the first paper. If we take $R = 2 \cdot 10^{12}$ (see art. 6, below), then for our sun we have $\mu = 10^{-20}$.

The remaining equation (36) is satisfied by

$$\alpha = \beta = -\gamma - \zeta.$$

The value of ζ is arbitrary. We can take $\zeta = 0$. We have thus, instead of (9A),

$$\lambda = \frac{1}{R^2}, \qquad \kappa\rho_0 = 2\lambda - \frac{3\mu}{R^2}.$$

For $r = \frac{1}{2}\pi R$ we have $\gamma = 0$; for larger values of r, which are only possible in spherical space, the numerical value of γ increases again; and for $r = 2\pi R$ we would have $\gamma = -\infty$, however small μ may be. In the elliptical space, of course, this difficulty does not exist, since distances exceeding $\frac{1}{2}\pi R$ are impossible.

In the system B we have, outside the sun, $\rho = 0$, and consequently also $p = 0$. The equations (33) then are dependent on each other. We now put

$$a = 1 + \alpha, \quad b = R^2 \sin^2\chi(1 + \beta), \quad f = \cos^2\chi(1 + \gamma),$$
$$\rho_0 = 0, \quad \lambda = 3\epsilon(1 + \zeta).$$

The equations become, to the first order,

$$(38) \quad \begin{cases} (a) & R^2\gamma'' + 2R\gamma'(\cot\chi - \tan\chi) + R\tan\chi(\alpha' - 2\beta') + 6(\alpha + \zeta) = 0, \\ (c) & (\beta - \alpha)\operatorname{cosec}^2\chi + 3(\alpha + \zeta) + R\beta'(\cot\chi - \tan\chi) + R\gamma'\cot\chi = 0. \end{cases}$$

These are satisfied by

$$\alpha = \frac{\mu}{\sin\chi} - \zeta,$$

$$\beta = -\frac{\mu}{\sin\chi} - \zeta,$$

$$\gamma = -\frac{\mu}{\sin\chi} + \zeta.$$

The value of ζ is irrelevant. We can take $\zeta = 0$. If we took $\zeta = \mu$, we would have $\alpha = \gamma = 0$ for $r = \frac{1}{2}\pi R$. If then we introduce

$$r' = r(1 - \tfrac{1}{2}\mu), \quad R' = R(1 - \tfrac{1}{2}\mu), \quad t' = t(1 + \tfrac{1}{2}\mu),$$

we have, neglecting the square of μ, for these new variables:

$$\chi = \frac{r'}{R'}, \qquad \lambda = \frac{3}{R'^2},$$

$$a' = 1 + \frac{\mu}{\sin\chi}, \qquad b' = R^2\sin^2\chi\left(1 - \frac{\mu}{\sin\chi}\right), \qquad f' = \cos^2\chi\left(1 - \frac{\mu}{\sin\chi}\right),$$

or the same formulas as for the unaccented letters with $\zeta = 0$.

6. We will now try to make some estimates of the value of R,

which must be adopted in order not to contradict the known data of observations. We will throughout use astronomical units: the unit of time being the day, of length the mean distance of the earth from the sun, and of mass the sun's mass.* In these units we have

$$c = 173, \quad k = \tfrac{1}{4} \cdot 10^{-6}.$$

We will first take the system A.

In elliptical space the apparent angular diameter of an object whose linear diameter is d, at the distance $r = R\chi$ from the earth, is

$$\delta = \frac{d}{R \sin \chi}.$$

It is very probable that at least some of the spiral nebulæ or globular clusters are galactic systems comparable with our own in size. If then we take for the diameters, *e.g.*, $d = 10^9$ and $\delta = 5'$, then even for the maximum distance $r = \frac{1}{2}\pi R$ we would have $R = 6 \cdot 10^{11}$. We are thus led to take, roughly,

$$R \geqq 10^{12}. \tag{39}$$

The total volume of elliptical space is $V_0 = \pi^2 R^3$. We have $\kappa\rho_0 = 2/R^2$, consequently the total mass of the world-matter is $M_0 = 2\pi^2 R/\kappa$, or

$$M_0 = 8 \cdot 10^7 R, \qquad \rho_0 = \frac{8 \cdot 10^6}{R^2}.$$

If for M_0 we took the mass of our galactic system, which may be estimated † at $\frac{1}{3} \cdot 10^{10}$, we would find $R = 41$, which, of course, is absurd. A better estimate is found if we start from ρ_0 and take for this the star-density at the centre of the galactic system,‡ which may be estimated at about 80 stars in a unit of volume of Kapteyn (1000 cubic parsecs), or $\rho_0 = 10^{-17}$ in our units. We then find

$$R = 9 \cdot 10^{11}. \tag{40}$$

* For the sake of comparison we may add that 10^6 astron. units = 5 parsecs = 16 light-years = $15 \cdot 10^{18}$ centimetres. The mass of the sun is $2 \cdot 10^{33}$ grams. A density 1 in astronomical units is therefore equivalent to $6 \cdot 10^{-7}$ in C.G.S. units.

† Communicated by Professor Kapteyn. The estimate is based on van Rhyn's recent investigation of the number of stars (*Groningen Publications*, 27), assuming the average mass of the stars to be the same as that of the sun.

‡ In his paper of 1900, which is quoted below, Schwarzschild considers an elliptical universe just large enough to contain our galactic system (with a constant density equal to the star-density near the sun). This would be a very simple solution of the problem of what prevents the disintegration of the galactic system by the proper motions of the stars. The same argument is used by Einstein in the introductory paragraph of his *Kosmologische Betrachtungen*. It is evident, however, from the numerical data given above, that the elliptical space of the system A does *not* fulfil this purpose. In it the stars can escape from the galactic system quite as easily as in the classical euclidean space (system C).

The total mass would then be $M_0 = 7 \cdot 10^{19}$, and the volume $V_0 = 7 \cdot 10^{36}$.

It is very probable that in the part of space which immediately surrounds our galactic system there are many similar systems whose mutual distances are large compared with their dimensions. If for the average shortest distance between neighbouring systems we take 10^{10}, and if further we suppose that not only our neighbourhood but the whole universe is thus filled with galactic systems, there would be room for $7 \cdot 10^6$ of such systems. If each of these had a mass of $\frac{1}{3} \cdot 10^{10}$, their combined mass would be $2 \cdot 10^{16}$, or only $\frac{1}{3000}$ of the total mass of the world-matter. According to this view, only a small portion of the world-matter would be condensed into ordinary matter. It is, however, very well possible to imagine a world in which all the world-matter would be thus condensed. We must then, as unit of volume with which to measure the average density ρ_0, take a space which is large with respect to the mutual distances of the galactic systems. With the numerical data adopted above, we would then have $\rho_0 = \frac{1}{3} \cdot 10^{-20}$, from which

$$R \leqq 5 \cdot 10^{13}. \tag{41}$$

I write the sign $\leqq$ instead of $=$ because, if we took a still larger value for R, the total mass of the world-matter would not be sufficient to fill the universe with galactic systems. We can thus consider the value (41) as an upper limit—subject, of course, to the uncertainty (which is considerable) of the hypotheses and of the numerical data from which it was derived.

Space being finite, and the straight line closed, we should, at the point of the heavens opposite the sun, see an image of the back side of the sun. This not being the case, the light must be absorbed on its way "round the world." Schwarzschild * estimates that an absorption of 40 magnitudes would be sufficient. If we accept the result of Shapley,† that the absorption in intergalactic space is less than 0·0001 mag. in a Kapteyn's unit of distance (10 parsecs), then for an absorption of 40 mags. to be produced in a distance of πR we must have, in our units,

$$R > \tfrac{1}{4} \cdot 10^{12}. \tag{42}$$

King ‡ has derived the density of matter in space from the coefficient of selective absorption. The selective absorption found by Shapley is about one-fiftieth of the value used by King. The latter finds a density of 6300 suns per cubic parsec. Shapley's absorption would thus require one-fiftieth of this, or $\frac{3}{2} \cdot 10^{-14}$ in our units. This would correspond to $R = 2 \cdot 10^{10}$. With this value of R the total absorption in the distance πR would only be 3·6 mags., or only one-eleventh of the required value. To get an

* "Ueber das zulässige Krümmungsmaass des Raumes," *Vierteljahrssch. der Astr. Ges.*, vol. xxxv. (1900), p. 337.

† *Mount Wilson Contributions*, No. 116.

‡ *Nature*, vol. xcv. p. 701.

absorption of 40 mags., we must multiply the density by 11^2, and consequently divide R by 11. This would give

$$\rho_0 = 2 \cdot 10^{-12}, \qquad R = 2 \cdot 10^9.$$

This density appears to be much too large, and R much too small, to be admissible. The two assumptions underlying this determination are: 1st, that the coefficient of general absorption (extinction) is equal to that of selective absorption by molecular scattering; and 2nd, that the world-matter consists of molecules of hydrogen. Both assumptions may be considerably in error, and the extinction produced by a given density may well be much larger, and consequently the density needed for a given extinction much smaller, than has been here assumed.

Each of the estimates (39), (40), (41), (42) of course is subject to a very large uncertainty. Their near agreement is rather remarkable, and could not have been expected *a priori*.

7. In the system B the rays of light are straight lines in hyperbolical space. The parallax has, according to the formula (20′), a minimum $p_0 = 1/R$. Schwarzschild has, in the paper already quoted, derived a lower limit for the value of R of hyperbolical space, from the fact that there are certainly stars with parallaxes equal to $0''{\cdot}05$ or smaller. He thus found

$$R > 4 \cdot 10^7.$$

Of course only actually measured absolute parallaxes can be taken into account. All parallaxes measured after 1900 are relative ones, and consequently the limit found by Schwarzschild still corresponds to our present knowledge.

The reasoning by which the value (39) of R was derived for the system A is not applicable to the system B, for the relation between apparent and linear diameter is here

$$\delta = \frac{d}{R \sinh \frac{h}{R}} = \frac{d}{R \tan \chi},$$

which shows that δ is zero for $r = \frac{1}{2}\pi R$. If we accept the existence of a number of galactic systems whose average mutual distances are of the order of 10^{10}, all we can say is that πR must be several times 10^{10}, or roughly

$$R > 10^{11}. \tag{43}$$

Also the estimate of R, based on the fact that we do not see the back of the sun, is not applicable to the system B, because light requires an infinite time for the "voyage round the world."

In the system B we have $g_{44} = \cos^2 \chi$. Consequently the frequency of light-vibrations diminishes with increasing distance from the origin of co-ordinates. The lines in the spectra of very distant stars or nebulæ must therefore be systematically displaced towards the red, giving rise to a spurious positive radial velocity.

It is well known that the helium stars do indeed show a systematic displacement, corresponding to about + 4·5 km./sec. If we ascribe about one-third of this to the mass of the stars themselves,* the rest, or + 3 km./sec., may be explained as an apparent displacement due to the diminution of g_{44}. For the average distance of the B-stars we can take † $r = R\chi = 3 \cdot 10^7$. We then have $1 - \cos\chi = 10^{-5}$, from which

$$R = \tfrac{2}{3} \cdot 10^{10}. \qquad (44)$$

Campbell has also found a systematic displacement of the same sign for the K-stars, whose average distance probably is the largest after the helium stars. For stars of other types both the systematic displacement and the average distance are smaller.

For the lesser Magellanic cloud Hertzsprung found the distance $r > 6 \cdot 10^9$. The radial velocity ‡ is about + 150 km./sec. This gives

$$R > 2 \cdot 10^{11}. \qquad (45)$$

The formulas (25′), for small values of r, become the same as in classical mechanics. For large values of r there is no reason why the angular proper motion $\frac{d\theta}{dt}$ should not decrease in the same way as it does in Newtonian mechanics. The total linear velocity, however, and consequently also the radial velocity, may on the average be expected to increase up to $\chi = \frac{1}{4}\pi$, owing to the first term on the right in the second formula (25′). We should thus, in the system B, for stars in our neighbourhood expect radial and transversal velocities of the same order, but for objects at very large distances we should expect a greater number of large or very large radial velocities. Spiral nebulæ most probably are amongst the most distant objects we know. Recently a number of radial velocities of these nebulæ have been determined. The observations are still very uncertain, and conclusions drawn from them are liable to be premature. Of the following three nebulæ, the velocities have been determined by more than one observer: §

Andromeda	(3 observers)	− 311 km./sec.
N.G.C. 1068	(3 ,,)	+ 925 ,,
N.G.C. 4594	(2 ,,)	+ 1185 ,,

These velocities are very large indeed, compared with the usual velocities of stars in our neighbourhood.

The velocities due to inertia, according to the formula (25′), have no preference of sign. Superposed on these are, however, the apparent radial velocities due to the diminution of g_{44}, which are positive. The mean of the three observed radial velocities stated

* See first paper, p. 719.
† See *Astrophysical Journal*, vol. xxxii. p. 90.
‡ Report to the Council, 1917 Feb., *M.N.*, lxxvii. p. 376.
§ Report to the Council, 1917 Feb., *M.N.*, lxxvii. pp. 375, 383.

above is +600 km./sec. If for the average distance we take 10^5 parsecs $= 2 . 10^{10}$, then we find

$$R = 3 . 10^{11}. \tag{46}$$

Of course this result, derived from only three nebulæ, has practically no value. If, however, continued observation should confirm the fact that the spiral nebulæ have systematically positive radial velocities, this would certainly be an indication to adopt the hypothesis B in preference to A. If it should turn out that no such systematic displacement of spectral lines towards the red exists, this could be interpreted either as showing A to be preferable to B, or as indicating a still larger value of R in the system B.

Doorn: 1917 *July.*

The Equations of Radiative Transfer of Energy.

By J. H. Jeans.

1. In a gaseous star it is probable that much more energy is transferred by radiation than by ordinary gaseous conduction, so that an accurate determination of the laws of radiative transfer is a necessary preliminary to many problems of stellar physics. In two recent papers * Professor Eddington has given an equation of transfer which I believe to be erroneous (at least when considered with reference to its application), and which has, I believe, led him to an erroneous result. In the present paper I have examined independently the laws of radiative transfer.

These laws, like those of ordinary conduction of heat, can of course be found in a general form independently of the special problem to which they are ultimately to be applied; we need not complicate the laws by introducing (as Professor Eddington does) the curvature of the star's figure.

Consider a medium arranged so that layers of equal temperature are normal to the axis of x, and consider the stream of radiation making an angle θ with this axis. At the plane $x = \xi$ let the stream of radiation crossing a plane of cross-section $d\sigma$ per unit time in directions contained within a small cone $d\omega$ be $I d\sigma d\omega$, where I is a function of x and θ.

After traversing a length of path ds, the main part of the radiation will reach the plane $x = \xi + ds \cos\theta$. On this path it will have been diminished in a ratio $1 - c\rho ds$, where c is a coefficient of opacity for this particular radiation, measured per unit mass, and ρ is the density of the medium. It will also have been reinforced by radiation emitted by the matter traversed. The volume of this matter is $ds d\sigma$, so that if E is the emission

* *M.N.*, 1916 Nov. and 1917 June.

per unit mass per unit time, the emission will be $E\rho ds d\sigma$, of which a fraction $d\omega/4\pi$ will lie within the cone $d\omega$.

Thus the beam, when it arrives at the plane $x = \xi + ds\cos\theta$, will be of intensity

$$I d\sigma d\omega(1 - c\rho ds) + E\rho ds d\sigma d\omega/4\pi,$$

whence we obtain, as the increment of I in passing over unit length of path,

$$\cos\theta \frac{\partial I}{\partial x} = \frac{E\rho}{4\pi} - c\rho I \quad . \quad . \quad . \quad . \quad (1)$$

This is the general equation of radiative transfer.

2. In a gas which is in equilibrium at uniform temperature $\partial I/\partial x$ of course vanishes, so that

$$E = 4\pi c I \quad . \quad . \quad . \quad . \quad (2)$$

I now being independent of θ.

In the general problem in which the temperature is not uniform, I is not independent of θ, the whole transfer of energy arising from the variations in I. Following Eddington, let us assume

$$I = A + BP_1 + CP_2 + \ldots \quad . \quad . \quad . \quad (3)$$

where P_1, P_2 . . . are Legendre's coefficients and A, B, C . . . are functions of x.

Equation (1) now assumes the form

$$\cos\theta\left(\frac{dA}{dx} + P_1\frac{dB}{dx} + P_2\frac{dC}{dx} + \ldots\right) = \frac{E\rho}{4\pi} - c\rho(A + BP_1 + CP_2 + \ldots) \quad (4)$$

Using the harmonic relation

$$(2n+1)P_n\cos\theta = (n+1)P_{n+1} + nP_{n-1},$$

this equation becomes

$$P_1\frac{dA}{dx} + \tfrac{1}{3}(2P_2 + 1)\frac{dB}{dx} + \tfrac{1}{5}(3P_3 + 2P_1)\frac{dC}{dx} + \ldots$$
$$= \frac{E\rho}{4\pi} - c\rho(A + BP_1 + CP_2 + \ldots).$$

Since this must be true for all values of θ, we may equate coefficients of the different harmonics, and obtain

$$\tfrac{1}{3}\frac{dB}{dx} = \frac{E\rho}{4\pi} - c\rho A \quad . \quad . \quad . \quad (5)$$

$$\frac{dA}{dx} + \tfrac{2}{5}\frac{dC}{dx} = -c\rho B \quad . \quad . \quad . \quad . \quad (6)$$

$$\tfrac{2}{3}\frac{dB}{dx} + \tfrac{3}{7}\frac{dD}{dx} = -c\rho C, \text{ etc.} \quad . \quad . \quad . \quad (7)$$

The total absorption of energy per unit volume per unit time is

$$\int\int \mathrm{I}c\rho d\omega = 2\pi c\rho \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} \mathrm{I} \sin\theta d\theta = 4\pi c\rho \mathrm{A}.$$

The total emission of the same unit volume has been assumed to be $\mathrm{E}\rho$. Let us assume that the energy of this emitted radiation comes partly from the re-emission of the absorbed radiation, but partly also from radioactive or similar sources which generate energy at a rate $4\pi\epsilon$ per unit mass. Then the equation of energy of the unit volume under consideration, divided throughout by ρ, is

$$4\pi c\mathrm{A} + 4\pi\epsilon = \mathrm{E} \quad . \quad . \quad . \quad . \quad (8)$$

Combining this with equation (5), we find

$$\tfrac{1}{3}\frac{d\mathrm{B}}{dx} = \epsilon\rho \quad . \quad . \quad . \quad . \quad (9)$$

The total net transfer of radiation per unit area per unit time across the plane $x = \xi$ is $\int\int \mathrm{I}\cos\theta d\omega$ or $\frac{4}{3}\pi\mathrm{B}$. Thus equation (9), which can be expressed in the form

$$\frac{d}{dx}(\tfrac{4}{3}\pi\mathrm{B}) = 4\pi\epsilon\rho,$$

is obviously the "equation of continuity" of energy.

Combining (7) with (9), we obtain

$$\tfrac{2}{7}\frac{d\mathrm{D}}{dx} + c\rho\mathrm{C} = -2\epsilon\rho \quad . \quad . \quad . \quad (10)$$

In Professor Eddington's two recent papers ϵ is supposed to be finite, while C, D are neglected as being small. Our exact equation (10) shows this procedure to be illegitimate. The error involved is of the order of magnitude of ϵ, which in turn is the order of magnitude of Professor Eddington's main result.

3. In the simplest and most natural case in which $\epsilon = 0$, equation (9) shows that B must be constant, and equations (5), (6), (7) now reduce to

$$\mathrm{E} = 4\pi c\mathrm{A} \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{d\mathrm{A}}{dx} + \tfrac{2}{5}\frac{d\mathrm{C}}{dx} = -c\rho\mathrm{B} \quad . \quad . \quad . \quad . \quad (12)$$

$$\tfrac{2}{7}\frac{d\mathrm{D}}{dx} = -c\rho\mathrm{C}, \text{ etc.} \quad . \quad . \quad . \quad (13)$$

For a uniform temperature gradient, or a gradient which may be treated as uniform over a range which is large compared with $1/c\rho$, $d\mathrm{A}/dx$, $d\mathrm{C}/dx$, etc., will all be constants. Hence the right-hand members of equations (12), (13), etc., are constants,

so that B, C, D . . . are constants. Thus equations (13) *et seq.* reduce to $C = D = E = \ldots = 0$, while equations (11) and (12) become

$$E = 4\pi c A \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{dA}{dx} = -c\rho B \quad . \quad . \quad . \quad . \quad (15)$$

If c denote the velocity of light, the energy of radiation per unit volume is

$$\frac{1}{c}\int\int I d\omega = \frac{4\pi}{c} A \quad . \quad . \quad . \quad . \quad (16)$$

and this may be put equal to aT^4 where a is the universal constant $7{\cdot}1 \times 10^{-15}$ C.G.S. units. Since $ac = 4\sigma$ where σ is Stefan's constant ($5{\cdot}32 \times 10^{-5}$), we have

$$\pi A = \sigma T^4 \quad . \quad . \quad . \quad . \quad (17)$$

so that, from equation (15), the net flow of radiant energy per unit area is

$$\tfrac{4}{3}\pi B = -\frac{16\sigma T^3}{3c\rho}\frac{\partial T}{\partial x}.$$

Thus the flow may be put in the usual form $-\kappa \partial T/\partial x$, where κ is a coefficient of radiative conduction. Its value in energy-units is

$$\kappa = \frac{16\sigma T^3}{3c\rho} \quad . \quad . \quad . \quad . \quad (18)$$

while its value in heat-units is

$$\kappa = \frac{16\sigma T^3}{3cJ\rho} = 6{\cdot}8 \times 10^{-12}\frac{T^3}{c\rho} \quad . \quad . \quad . \quad (19)$$

4. In the interior of a star T will be of the order of 10^6, and for a giant star we may assign to ρ a value of the order of 10^{-2}. Thus κ is of the order of $10^9/c$.

At 0° C. the coefficient of ordinary gaseous conduction of heat is of the order of 5×10^{-5} for CO_2 and gases of similar molecular weight. Its increase is at most proportional to the absolute temperature, so that at 10^6 degrees its value will be at most of the order of 0·2. Clearly, then, whatever reasonable value we assign to c, gaseous conduction will be quite insignificant in comparison with radiative conduction.

In a gas at the high temperature of the interior of a star there must be a great amount of ionisation, so that electronic conductivity must in any case be expected to outweigh gaseous conductivity. It is very difficult to form an estimate of the degree of conductivity under such conditions, but various lines of reasoning (*e.g.* comparison with metallic conduction, etc.) agree in suggesting that ordinary conductivities of all kinds are insignificant in comparison with radiative conductivity in a star's interior.

5. The simple solution found in § 3, namely,

$$I = A + B \cos \theta,$$

will be accurate in the interior of a star, but will fail as the surface is approached. For at the surface there must be no entrant flow of radiation, so that the true value of I must vanish for all values of θ between 90° and 180°; there will, moreover, be a discontinuity in the value of I when $\theta = 90°$. It will be found that an expansion of I in harmonics is unsuitable for the investigation of the flow of energy near the boundary, such an expansion becoming divergent as the boundary is approached.

We accordingly return to the fundamental equation (1). Write μ for $\cos \theta$, and put

$$\int c\rho dx = \nu,$$

making $\nu = 0$ at the boundary. Thus ν is negative inside the star, $-\nu$ measuring the aggregate absorbing power of the matter between an interior point and the star's surface. The fundamental equation (1) now assumes the form

$$\mu \frac{\partial I}{\partial \nu} = \frac{E}{4\pi c} - I \quad . \quad . \quad . \quad . \quad (20)$$

This equation is true for all values of μ from -1 to $+1$. Multiply by $d\mu$ and integrate through this range and we obtain

$$\frac{\partial}{\partial \nu} \int_{-1}^{+1} \mu I d\mu = \frac{E}{2\pi c} - \int_{-1}^{+1} I d\mu \quad . \quad . \quad . \quad (21)$$

Here $2\pi \int_{-1}^{+1} \mu I d\mu$ measures the net transfer of energy across a plane parallel to the axis of x, so that the left-hand member is equal to $2\epsilon/c$. If, as before, we take $\int_{-1}^{+1} I d\mu$ to be equal to $2A$, equation (21) is readily seen to be identical with our former equation of energy (8), while equation (20) becomes

$$\mu \frac{\partial I}{\partial \nu} = A + \frac{\epsilon}{c} - I \quad . \quad . \quad . \quad . \quad (22)$$

Consider for the present only the case of $\epsilon = 0$. The simplest solution is that already discussed, namely,

$$I = A + B\mu, \quad A = A_0 - \nu B; \quad . \quad . \quad . \quad (23)$$

the general solution is found to be

$$I = e^{-\nu/\mu} \left[\int^{\nu} e^{\nu/\mu} A \frac{d\nu}{\mu} + \phi(\mu) \right] \quad . \quad . \quad . \quad (24)$$

in which $\phi(\mu)$ is an arbitrary function of μ. To investigate the flow of radiation near the boundary we require to find a solution

of the type (24), which shall reduce to (23) in the far interior of the star ($\nu = -\infty$). Let us suppose the required solution to be

$$A = A_0 - \nu B + C,$$

where C is a function of ν only, which vanishes when $\nu = -\infty$.

Inserting this value for A into equation (24) and simplifying, we find

$$I = A_0 - \nu B + \mu B + e^{-\nu/\mu}\left[\int^{\nu} e^{\nu/\mu} C \frac{d\nu}{\mu} + \phi(\mu)\right] \quad . \quad (25)$$

Since $\phi(\mu)$ is already at our disposal, the lower limit of integration is also at our disposal. To keep the integrals convergent, we must take the lower limit to be $\nu = 0$ when μ is negative and $\nu = -\infty$ when μ is positive. We may accordingly suppose the value of I at $\nu = \nu_0$ to be

$$I = A_0 - \nu_0 B + \mu B + e^{-\nu_0/\mu}\left[\int_{\infty}^{\nu_0} e^{\nu/\mu} C \frac{d\nu}{\mu} + \phi(\mu)\right] \quad . \quad (26)$$

when μ is positive (emergent wave) and

$$I = A_0 - \nu_0 B + \mu B + e^{-\nu_0/\mu}\left[\int_{0}^{\nu_0} e^{\nu/\mu} C \frac{d\nu}{\mu} + \psi(\mu)\right] \quad . \quad (27)$$

when μ is negative (entrant wave).

The values of $\phi(\mu)$ and $\psi(\mu)$ must now be determined from the boundary conditions. In the far interior of the star ($\nu = -\infty$) the emergent wave must be unaffected by the presence of the boundary, so that $\phi(\mu) = 0$. At the surface of the star the entrant wave must be of zero intensity for all negative values of μ, so that

$$\psi(\mu) = -(A_0 + \mu B).$$

This completes the evaluation of I in terms of C; it remains to determine C, which can be evaluated from the relation

$$A = \tfrac{1}{2}\int_{-1}^{+1} I d\mu \quad . \quad . \quad . \quad . \quad (28)$$

Inserting the values for A and I, this relation becomes

$$A_0 - \nu_0 B + C = A_0 - \nu_0 B + \tfrac{1}{2}\int_0^1 e^{-\nu_0/\mu}\left[\int_{-\infty}^{\nu_0} e^{\nu/\mu} C \frac{d\nu}{\mu}\right] d\mu + \tfrac{1}{2}\int_{-1}^{0} e^{-\nu_0/\mu}\left[\int_0^{\nu_0} e^{\nu/\mu} C \frac{d\nu}{\mu} - A_0 - \mu B\right] d\mu.$$

On rearranging the integrals, this becomes

$$C = \tfrac{1}{2}\int_{-\infty}^{\nu_0} C \int_0^1 e^{\frac{\nu-\nu_0}{\mu}} \frac{d\mu}{\mu} d\nu + \tfrac{1}{2}\int_0^{\nu_0} C \int_{-1}^{0} e^{\frac{\nu-\nu_0}{\mu}} \frac{d\mu}{\mu} d\nu - \tfrac{1}{2}\int_{-1}^{0} e^{-\nu_0/\mu}(A_0 + \mu B) d\mu \quad (29)$$

This integral equation will determine C exactly, but I have not succeeded in solving it except by approximation.

6. A first approximation is clearly

$$C = -\tfrac{1}{2}\int_{-1}^{0} e^{-\nu_0/\mu}(A_0 + \mu B)d\mu \quad . \quad . \quad . \quad (30)$$

We shall examine this first, and consider the error of the approximation later. The value of C at the boundary ($\nu_0 = 0$) is

$$C = -\tfrac{1}{2}(A_0 - \tfrac{1}{2}B) \quad . \quad . \quad . \quad . \quad (31)$$

and in the general value of C each term falls off to zero in the interior of the star.

At the boundary there is no entrant wave ($I = 0$), while from equation (26) the emergent wave is given by

$$I = A_0 + \mu B + C'$$

where

$$C' = \int_{-\infty}^{0} e^{-\nu/\mu} C \frac{d\nu}{\mu}.$$

If C were constant for all values of ν, the value of C′ would be identical with that of C, but as C varies, C′ must be regarded as an average value of values of C taken some distance into the star. When $\mu = 0$, C′ becomes identical with the value of C at the boundary; for other values of μ, C′ is less than this.

At the boundary the general relation (28) becomes

$$\begin{aligned} A_0 + C &= \tfrac{1}{2}\int_0^1 I d\mu \\ &= \tfrac{1}{2}A_0 + \tfrac{1}{4}B + \tfrac{1}{2}\int_0^1 C' d\mu, \end{aligned}$$

in which the last term is rather less than the value of $\frac{1}{2}C$ at the boundary, say $\frac{1}{2}\eta C$, where $\eta < 1$. Thus the equation becomes

$$C(1 - \tfrac{1}{2}\eta) = -\tfrac{1}{2}(A_0 - \tfrac{1}{2}B) \quad . \quad . \quad . \quad (32)$$

This is consistent with (31) only if

$$A_0 = \tfrac{1}{2}B \quad . \quad . \quad . \quad . \quad (33)$$

in which case $C = 0$ at the boundary. It now follows from equation (30) that C is small for all values of ν, and hence that equation (30) gives a good approximation to the true solution of equation (29).

One equation remains, namely,

$$\int_{-1}^{+1} I\mu d\mu = \tfrac{2}{3}B.$$

At the surface this assumes the form

$$\int_0^1 (A_0 + \mu B + C')\mu d\mu = \tfrac{2}{3}B$$

or

$$\tfrac{1}{2}A_0 + \tfrac{1}{3}B + \tfrac{1}{2}C'' = \tfrac{2}{3}B,$$

where C'' is another average value of C. This becomes

$$C'' = -(A_0 - \tfrac{2}{3}B),$$

and is again satisfied approximately by equation (33).

7. Thus a solution which approximately satisfies all the necessary conditions is $C = 0$, $A_0 = \frac{1}{2}B$. The value of the temperature is given by (*cf.* equation (17)).

$$\sigma T^4 = \pi A = \pi(A_0 - \nu B) = \pi B(\tfrac{1}{2} - \nu) \quad . \quad . \quad (34)$$

In this approximate solution it appears that $\partial A/\partial \nu = -B$ right up to the boundary, and this equation is identical with equation (15), which was found to be exactly true in the star's interior. Thus the results given in § 3, which are accurately true inside the star, are also approximately true up to the surface, and even in the outermost layers there may be supposed to be a radiative flow of heat $-\kappa \partial T/\partial x$ where κ is given by equations (18) and (19).

From equations (26) and (27) the stream of radiation at any internal point is given by

$$I = (\tfrac{1}{2} - \nu + \mu)B \quad . \quad . \quad . \quad . \quad (35)$$

when μ is positive (emergent wave), and

$$I = [(\tfrac{1}{2} + \mu)(1 - e^{-\nu/\mu}) - \nu]B \quad . \quad . \quad . \quad (36)$$

when μ is negative (entrant wave).

At the surface ($\nu = 0$) the entrant wave is zero for all values of μ, while the emergent wave is given by

$$I = (\tfrac{1}{2} + \cos\theta)B \quad . \quad . \quad . \quad . \quad (37)$$

This last formula ought of course to give the darkening at a star's limb, θ being the angle between the normal to the surface and the line of sight. It is at once seen to be of the same form as the ordinarily assumed formula *

$$I = I_0(1 - x + x\cos\theta)$$

and agrees with this if x, the "coefficient of darkening," is taken equal to $\frac{2}{3}$. This is about the actual value found for the sun by Schwarzschild and Abbot.

8. The total outward flow of energy per unit area is $\frac{4}{3}\pi B$. If this is regarded as coming from a photosphere at temperature T_1, we must have

$$\sigma T_1^4 = \tfrac{4}{3}\pi B \quad . \quad . \quad . \quad . \quad (38)$$

* *Cf.* Shapley, *Orbits of Eclipsing Binaries* (Princeton, 1915), p. 2.

so that, by comparison with equation (34), the position of the photosphere will be given by $-\nu = \frac{5}{6}$, or, replacing ν by its value,

$$\int \rho dx = \tfrac{5}{6} \quad . \quad . \quad . \quad . \quad (39)$$

where the integral is taken from the boundary to the photosphere.

From equation (34) the temperature T_0 at the outer surface is given by

$$\sigma T_0^4 = \tfrac{1}{2}\pi B \quad . \quad . \quad . \quad . \quad (40)$$

so that, from equations (38) and (40), $T_1^4/T_0^4 = \frac{8}{3}$, or $T_1 = 1{\cdot}28T_0$.

If, however, we consider only the radiation which leaves the star normally, a different result is obtained. The normal stream of radiation per unit area is, from equation (37), $I = \frac{3}{2}B$, and if this is regarded as coming from a photosphere at temperature T_2, we have

$$\sigma T_2^4 = \tfrac{3}{2}\pi B,$$

giving, by comparison with (34), $\nu = -1$ and $T_2 = 1{\cdot}32T_0$.

The light coming from the limb, naturally enough, may be regarded as coming from a photosphere at zero depth and at temperature T_0.

9. To avoid any possibility of misunderstanding, it may be well to repeat in conclusion that equations (34)–(39) are only approximations. There would not, I think, be any serious trouble in extending these approximations to higher orders of accuracy, but it seems probable that the gain in accuracy would only be small compared with the error involved in the neglect of scattering.

The Evolution and Radiation of Gaseous Stars.

By J. H. Jeans.

1. In a well-known lecture, "Relations between the Spectra and other Characteristics of the Stars,"* Professor H. N. Russell gave a diagram in which the absolute brightness of giant stars was seen to be approximately independent of their spectral type. One possible interpretation—although, of course, not the only one—is that the total emission of light of these stars does not change as their evolution progresses.

If this is the true interpretation, it is of interest to inquire whether the result is universally true, or is true only of certain stars. If the law expresses some general property of matter or of stellar structure, we should expect it to extend throughout the universe. On the other hand, it may be only a peculiarity of stars of our own system or near us—Russell's stars all being selected for well-determined parallax,—in which case other laws may be expected to obtain in other parts of space. Shapley has recently found that, of the giant stars in the Hercules cluster, the

* *Popular Astronomy*, xxii. 5 and 6.

red stars are the brightest, and that there is a steady decrease of luminosity from red to blue.

2. In recent communications to the *Monthly Notices* and to *Nature*, Professor Eddington has maintained that the uniformity of brightness arises from fundamental laws of radiation; he claims to have proved that the total radiation from a giant star depends only on its mass, and so is independent of its stage of evolution. Except for a complication arising from ionisation, he believes the radiation from a giant star of mass M to be of the form KM where K is a constant.

Professor Eddington nowhere states explicitly the value of K, but it can be deduced from his formulæ. From equations (25) and (29) of his first paper,* K is at once found to be $4\pi\epsilon$, so that the total emission of radiation is $4\pi M\epsilon$. But on turning back in his paper to find the meaning of ϵ, it appears that ϵ is defined to be such that the rate of generation of energy of the whole star is assumed at the outset to be $4\pi M\epsilon$.† Thus the result obtained is right but obvious, and the whole discussion does not advance our knowledge of stellar radiation in the least. This result being indisputably right, any different result such as that obtained in a second paper‡ must, so far as I can see, be necessarily wrong. In the paper preceding this I have tried to show where at least one error enters into this second paper of Professor Eddington.

3. It is surely obvious without discussion that there can be no perfectly general laws of the type enunciated by Professor Eddington for stars supposed to be in a steady state. The rate of emission of energy, being also the rate of generation of energy in the star's interior, must depend on the ultimate source of this energy. Clearly a star made of imaginary matter a million times more radioactive than uranium will emit energy at a greater rate than a non-radioactive star of the same mass. Hence a preliminary to any attack on the general problem must be a decision as to the source of energy. Many investigations§ have shown that the generation of energy arising from radioactivity in a star can be at most only a small fraction of that produced by gravitational contraction. More recently|| I have given reasons for believing the same to be true generally of all electrical and atomic sources of energy. It accordingly appears probable that a fair approximation to actual conditions will be obtained by regarding the star as a mass of gas contracting under its own gravitation, and having no sources of energy except those of gravitational contraction.

In the present paper I have given the outline of an investigation of the course of events as a mass of gas contracts under its own gravitation.

* *M.N.*, 1916 Nov.
† *Loc. cit.*, p. 10.
‡ *M.N.*, 1917 June.
§ See, in particular, F. A. Lindemann, *Nature*, 1915 April 22.
|| *Nature*, 1917 Aug. 2.

4. Let us fix our attention on a spherical mass of gas of radius a. Let T, p, ρ denote the temperature, pressure, and density at a distance r from the centre, and let the density ρ be supposed so small that these are connected by the ideal gas-equation

$$p = \frac{\mathrm{R}}{m}\rho\mathrm{T} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here R is the universal gas-constant, and m is the mass of a molecule of gas. When the gas consists of a mixture of different types of molecules, m may be supposed defined by equation (1).

Let M be the total mass of gas, and let M_r stand for

$$4\pi\int_0^r \rho r^2 dr,$$

the total mass inside a sphere of radius r. Then the equation of mechanical equilibrium is

$$\frac{dp}{dr} = -\frac{\mathrm{M}_r}{r^2}\rho \quad . \quad . \quad . \quad . \quad (2)$$

Let γ denote the ratio of the specific heats of the gas, which we shall assume to be uniform throughout, and let k be defined by the equation

$$p = k\rho^\gamma \quad . \quad . \quad . \quad . \quad . \quad (3)$$

so that k will in general vary from point to point of the gas. Combining this with equation (1), we find

$$\mathrm{T} = \frac{m}{\mathrm{R}}k\rho^{\gamma-1} \quad . \quad . \quad . \quad . \quad (4)$$

These are the fundamental equations, in which it will be noticed that r enters only in equation (2). It is convenient to replace r by a new co-ordinate q, equal to r/a, so that q varies from 0 at the centre to 1 at the surface of the mass of gas. Changing to this co-ordinate, equation (2) becomes

$$aq^2\frac{dp}{dq} = -\mathrm{M}_q\rho \quad . \quad . \quad . \quad . \quad (5)$$

where M_q denotes the mass inside a sphere of radius aq.

Homologous Configurations.

5. Now imagine our mass of gas to shrink uniformly through out into a sphere of radius a', so that the value of q remains unaltered for each particle of gas. Let p', ρ', T′ be the new values of p, ρ, and T.

From geometrical considerations we must have

$$\rho' a'^3 = \rho a^3 \quad . \quad . \quad . \quad . \quad (6)$$

The new configuration may or may not be a configuration of equilibrium. From equation (5) we see that the condition that it shall be a figure of equilibrium is that

$$a'p'/\rho' = ap/\rho,$$

or, from equations (1) and (6), that

$$a'\mathrm{T}' = a\mathrm{T} \quad . \quad . \quad . \quad . \quad (7)$$

This last equation expresses what is somewhat vaguely known as Lane's law. From equations (1), (6), and (7) we deduce

$$\frac{p'}{\mathrm{T}'^4} = \frac{p}{\mathrm{T}^4} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and again from equation (3)

$$k'a'^{(4-3\gamma)} = ka^{(4-3\gamma)} \quad . \quad . \quad . \quad . \quad (9)$$

A series of configurations of equilibrium such that ρ', T', p', and k' are connected with the corresponding quantities in a standard configuration by relations (6)–(9) may be called a series of homologous configurations.

6. The following theorems are easily proved :—

For a given mass of gas an equilibrium configuration is fully and uniquely determined by the value of a and the values of k at different distances from the centre.

Let the value of k be put in the form

$$k = a^{(3\gamma-4)}f(q),$$

then all configurations are got by varying a and $f(q)$. The configurations obtained by varying a while $f(q)$ remains the same form a homologous series. Different homologous series are accordingly represented by different functions of q.

Configurations for which $f(q)$ becomes infinite or negative at any point are physically impossible. Besides this, some values of $f(q)$ will represent configurations having a point mass at the centre. It is only configurations in which such a point mass does not figure that are of astronomical interest.

If a system loses energy by radiation and is constrained to move along a homologous series, k will vary for each element as $a^{(3\gamma-4)}$. Hence a mass will contract with radiation if $\gamma > \frac{4}{3}$, but will expand if $\gamma < \frac{4}{3}$. All configurations for which $\gamma < \frac{4}{3}$ are mechanically unstable, since energy can be gained by allowing them to expand to a state of infinite diffusion.

All configurations for which $\partial k/\partial r$ is negative at any point (or for which $\partial f(q)/\partial q$ is negative for any value of q between 0 and 1) are mechanically unstable. For energy can be gained by allowing two layers to change place convectively, if two can be found such that k is less for the outer than for the inner. In any configuration in which $\partial k/\partial r$ is negative over any range, convective currents

will be set up and the various layers will change places until a new configuration is formed in which $\partial k/\partial r$ will be positive or zero everywhere. If steady agencies are at work tending to depress the value of $\partial k/\partial r$ over any range to a value less than zero, a steady system of convection currents will be set up. If the viscosity of these currents is neglected, this system of currents will be of a strength just sufficient to prevent $\partial k/\partial r$ from falling below zero; the value of $\partial k/\partial r$ will be kept permanently equal to zero over the range in question, so that k will be constant and the equilibrium will be adiabatic.

7. Starting from any given initial configuration, we can calculate the changes produced in a short interval of time dt by conduction, radiation, contraction, etc., in the following manner:—

Imagine that each element of the gas is held at rest, and let the natural flow of heat take place for an interval dt, the temperature of the different elements being of course changed thereby, and therefore their pressures also. Next imagine that heat is temporarily prevented from flowing through the gas, so that each element can only change adiabatically, and let the different elements arrange themselves in equilibrium under their own gravitation. Then the final configuration will be identical with that which would naturally be reached after a time dt.

During the first process in which heat flows while the gas remains at rest, there will be a radial flow of heat $-\kappa\partial T/\partial r$ per unit area where κ is a coefficient which must be positive to conform with the second law of thermodynamics. Thus the aggregate flow of heat across a sphere of radius r, say E_r, will be

$$E_r = -4\pi\kappa r^2\frac{\partial T}{\partial r} \quad . \quad . \quad . \quad . \quad (10)$$

and the change of temperature will be given by

$$r^2\rho C_v\frac{\partial T}{\partial t} = \frac{\partial}{\partial r}\left(\kappa r^2\frac{\partial T}{\partial r}\right) \quad . \quad . \quad . \quad (11)$$

In this flow of heat ρ is kept constant, so that the change in T is connected with the corresponding change in k by the equation

$$\frac{1}{k}\frac{\partial k}{\partial t} = \frac{1}{T}\frac{\partial T}{\partial t} \quad . \quad . \quad . \quad . \quad (12)$$

During the second part of the motion the elements change adiabatically, so that k is not altered. Thus the whole rate of change of k is given by equations (11) and (12), or, in a single equation,

$$\frac{1}{k}\frac{dk}{dt} = \frac{1}{r^2\rho C_v T}\frac{\partial}{\partial r}\left(\kappa r^2\frac{\partial T}{\partial r}\right) \quad . \quad . \quad . \quad (13)$$

8. Starting from a given configuration, this equation enables us to determine the configuration after a time dt. It is, of course,

possible that the new configuration will be homologous to the original one. The condition for this is, from equation (9), that

$$\frac{1}{k}\frac{dk}{dt} = (3\gamma - 4)\frac{1}{a}\frac{da}{dt} = -\zeta \text{ (say)} \quad . \quad . \quad (14)$$

where ζ is constant through the mass, and is clearly always positive.

Integrating equation (13), we obtain, with the use of equations (10) and (14)

$$E_r = \zeta \int^r C_v T(4\pi\rho r^2) dr \quad . \quad . \quad . \quad (15)$$

Denote the integral on the right by H_r, this representing the total heat inside a sphere of radius r, then the condition for homologous contraction becomes

$$E_r = \zeta H_r . \quad . \quad . \quad . \quad . \quad (16)$$

At the boundary E_r must become identical with E, the total rate of emission of energy by radiation, while H_r becomes identical with H, the total heat-content of the mass. We accordingly have

$$E = \zeta H,$$

and on elimination of the unknown quantity ζ we obtain, as the condition for homologous contraction,

$$\frac{E_r}{E} = \frac{H_r}{H};$$

or again,

$$-4\pi\kappa r^2 \frac{\partial T}{\partial r} = \frac{H_r}{H} E \quad . \quad . \quad . \quad . \quad (17)$$

If κ has values such that this equation is satisfied at every point, the contraction will be homologous.

The Permanent Homologous Series.

9. Let us examine whether it is possible for a gas to contract naturally through a complete series of homologous configurations, say $k = a^{(3\gamma-4)} f(q)$.

For this to happen, equation (17) must be satisfied for all values of q and for all values of a. This equation may be regarded as determining κ for all values of q and a.

Assume that κ has such a value that homologous contraction does take place continually, so that equation (17) is satisfied for all values of a and q. We notice that, as contraction takes place, $r^2\partial T/\partial r$ and H_r/H do not depend on a, and so depend only on q, while E of course depends only on a. It follows that, for all values of q and a, κ must necessarily be of the form

$$\kappa = (\text{a function of } a) \times (\text{a function of } q) \quad . \quad . \quad (18)$$

In general, the transfer of heat will consist of two parts, one arising from conduction and one from radiation. Thus κ will be the sum of two separate terms, and so cannot be a simple product as demanded by equation (18). Thus in the general problem there can be no permanent homologous series.

We have, however, seen in the paper immediately preceding this, that in a giant star the transfer of heat by conduction will be almost negligible in comparison with the radiative transfer. Thus for a giant star κ may be supposed to be the coefficient of radiative transfer, of which the value was found to be

$$\kappa = \frac{16\sigma}{3c\rho}\mathrm{T}^3 \qquad . \quad . \quad . \quad . \quad (19)$$

In an actual problem the opacity c will depend to some extent on the temperature. To simplify the problem, let us suppose for the present that c is constant. Then κ is proportional to T^3/ρ, and this is unaltered by homologous contraction. Thus κ is a function of q only; condition (18) is satisfied, and we see that there can be a permanent homologous series for giant stars.

Substituting the value just given for κ into equation (17), we see that the permanent homologous series is defined by the equation

$$-\frac{64\pi\sigma}{3c}\frac{\mathrm{T}^3}{\rho}r^2\frac{\partial \mathrm{T}}{\partial r} = \frac{\mathrm{H}_r}{\mathrm{H}}\mathrm{E} \qquad . \quad . \quad . \quad (20)$$

Close to the surface of the star the equation is satisfied automatically, for, as we saw in the preceding paper, the radiative flow of energy (represented by the left-hand member of equation (20)) necessarily becomes equal to the surface emission E where $r=a$. At internal points equation (20) may be regarded as determining the temperature gradient $\partial \mathrm{T}/\partial r$ in terms of E, which in turn depends on the temperature gradient at the surface.

10. In equation (20) the quantities T^3/ρ, $r^2\partial \mathrm{T}/\partial r$ and H_r/H all remain unaltered by homologous contraction. It follows that on the permanent homologous series E is unaltered by contraction; for a mass of gas on this series the emission of radiation remains always the same.

As in the preceding paper, the total emission of radiation E may, to a rough approximation, be regarded as the emission from a definite photosphere. As the star contracts the depth of the photosphere below the surface will naturally diminish; on account of the lateral contraction of surface layers, this diminution will be more rapid than that of a. Thus the different photospheres will not form homologous points.

The position of the photosphere will be determined by the condition that the mass per unit area between it and the surface of the star is always the same quantity μ, μ being, as we saw in the preceding paper, about five-sixths of the mass necessary to reduce the intensity of a beam of light to a fraction $1/e$. Thus the

pressure at the photosphere will be μg, where g is the surface-value of gravity, and the temperature will be given by

$$\frac{\mathrm{R}}{m_1}\mathrm{T}_1\rho_1 = \mu g \quad . \quad . \quad . \quad . \quad . \quad (21)$$

quantities with suffix 1 referring to the photosphere.

The total emission of radiation is

$$\mathrm{E} = 4\pi r_1^2 \sigma \mathrm{T}_1^4 = \frac{4\pi\sigma \mathrm{M}\mu}{\mathrm{R}} \frac{m_1 \mathrm{T}_1^3}{\rho_1} \quad . \quad . \quad . \quad (22)$$

On the permanent homologous series E must, as we have seen, remain unaltered by contraction, so that $m_1\mathrm{T}_1^3/\rho_1$ must remain unaltered by contraction. In a homologous contraction T^3/ρ remains unaltered for homologous points, and m (the mean molecular mass at a point) may be supposed to remain approximately unaltered. In a mixture of gases there must necessarily be some rearrangement of the constituents when the gas contracts, and also, if there is any ionisation, the increase of temperature consequent on contraction must alter the degree of ionisation. These complications prevent strictly homologous contraction occurring at all, but one may, as an approximation, neglect them, and suppose that the mass contracts homologously, so that m remains always the same function of q.

Assuming this, $m\mathrm{T}^3/\rho$ will remain unaltered for homologous points by contraction, and the condition that $m_1\mathrm{T}_1^3/\rho_1$ (for the photosphere) shall remain unaltered by contraction, is that this quantity shall have a uniform value throughout the range within which the photosphere moves.

This range will be infinitesimal in comparison with the radius of the star, so that the required condition may be put in the form

$$\frac{d}{dr}\left(\frac{m\mathrm{T}^3}{\rho}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad (23)$$

at the boundary.

This equation, in conjunction with the equation of mechanical equilibrium, will determine the configuration of the figure near its surface; the arrangement of the interior layers will be determined by equation (20). In this way the configuration may be completely determined.

Mechanical Stability.

11. Let us next examine whether the configurations on the permanent series are mechanically stable.

We may begin with the outermost layers, for which the arrangement is determined by equation (23).

From the fundamental equations connecting p, ρ, T, and k we find that

$$\frac{m\mathrm{T}^3}{\rho} = \frac{m^4}{\mathrm{R}^3} k^{4/\gamma} p^{(3-4/\gamma)} \quad . \quad . \quad . \quad (24)$$

A condition for mechanical stability is that $3-4/\gamma$ shall be positive, so that $p^{(3-4/\gamma)}$ will increase as we pass inwards. Hence, to satisfy equation (23), km^{γ} must decrease as we pass inwards. If there is no ionisation, m will either be uniform in value (the only case in which strict homologous contraction is possible) or will increase as we pass inwards, so that $\partial k/\partial r$ will be positive. Thus the condition for mechanical stability is satisfied, and no convection currents will be set up to maintain stability near the surface. If, however, there is ionisation, m may decrease as we pass inwards, and convection currents may be set up near the surface.

12. We now turn to the consideration of the inner layers.

By differentiation of equation (4) we find

$$\frac{\partial T}{\partial r}=\frac{m}{R}\left[k(\gamma-1)\rho^{\gamma-2}\frac{\partial\rho}{\partial r}+\rho^{\gamma-1}\frac{\partial k}{\partial r}+\rho^{\gamma-1}\frac{k}{m}\frac{\partial m}{\partial r}\right],$$

while the equation of mechanical equilibrium (2) can be put in the form

$$k\gamma\rho^{\gamma-2}\frac{\partial\rho}{\partial r}+\rho^{\gamma-1}\frac{\partial k}{\partial r}=-\frac{M_r}{r^2}.$$

Eliminating the term in $\partial\rho/\partial r$, we find

$$-\frac{\partial T}{\partial r}=\frac{m}{R}\left[\frac{\gamma-1}{\gamma}\frac{M_r}{r^2}-\frac{1}{\gamma}\rho^{\gamma-1}\left(\frac{\partial k}{\partial r}+\frac{\gamma k}{m}\frac{\partial m}{\partial r}\right)\right] \quad . \quad . \quad (25)$$

Equation (21) determines the permanent homologous series. In it we may insert the value of $\partial T/\partial r$ just found, and also the value of the emission E given by equation (22). We find that the equation becomes

$$\frac{m_1T_1^3}{\rho_1}=\frac{mT^3}{\rho}\left(\frac{16(\gamma-1)}{3\mu c\gamma}\right)\left(\frac{M_r}{M}\right)\left(\frac{H}{H_r}\right)\left[1-\frac{r^2\rho^{\gamma-1}}{(\gamma-1)M_r}\left(\frac{\partial k}{\partial r}+\frac{\gamma k}{m}\frac{\partial m}{\partial r}\right)\right] \quad (26)$$

Here μ is the mass per unit area between the surface and the photosphere. In § 8 of the preceding paper we found the value of μc to be approximately $\frac{5}{6}$, so that the factor $16(\gamma-1)/3\mu c\gamma$ is always greater than unity; call it β. The factor M_rH/MH_r may be replaced by $\bar{T}/\bar{T}_r$, the ratio of the mean temperature of the whole mass to that inside a sphere of radius r. This is equal to unity near the surface, but decreases as we pass inwards. Equation (26) now assumes the form

$$1=\left(\frac{m}{m_1}\right)^4\left(\frac{k}{k_1}\right)^{4\gamma}\left(\frac{p}{p_1}\right)^{(3-4/\gamma)}\beta\frac{\bar{T}}{\bar{T}_r}\left[1-\frac{r^2\rho^{\gamma-1}}{(\gamma-1)M_r}\left(\frac{\partial k}{\partial r}+\frac{\gamma k}{m}\frac{\partial m}{\partial r}\right)\right] \quad (27)$$

in which quantities with suffix 1 refer to the photosphere, and others refer to a layer so far inside the star that the disturbing influence of the boundary may be neglected (with reference to actual conditions, this probably means a distance of only a few kilometres inside the star).

We need only discuss cases in which $3-4/\gamma$ is positive, other

cases being already unstable. For these $(p/\rho_1)^{(3-4/\gamma)}$ will increase as we pass inside the star, but $\bar{\mathrm{T}}/\bar{\mathrm{T}}_r$ will decrease. Except when $3-4/\gamma$ is very small, we may assume that the product of these quantities will increase as we pass inwards, for the former will change rapidly and the latter only slowly. Let us denote β times the product of these quantities by a new symbol α.

In the simplest case in which m has a uniform value throughout, equation (27) becomes

$$1 = \alpha\left(\frac{k}{k_1}\right)^{4/\gamma}\left[1 - \frac{r^2\rho^{\gamma-1}}{(\gamma-1)\mathrm{M}_r}\frac{\partial k}{\partial r}\right] \quad . \quad . \quad (28)$$

and this equation will determine the sequence of values of k on the permanent homologous series.

Let us examine whether $\partial k/\partial r$ can vanish at any point. If it does, the value of the factor $(k/k_1)^{4/\gamma}$ will be stationary at this point, while α will increase as we pass inwards. It follows that the factor in square brackets must increase as we pass outwards, so that, on passing outwards through this point, $\partial k/\partial r$ must change from positive to negative. But we have already seen that $\partial k/\partial r$ must be positive at the surface. It follows that $\partial k/\partial r$ can never change sign; it will be positive everywhere, and so the configuration will be stable without convection currents being set up.

It is seen by a similar argument that an increase in m as we pass inwards will add to the stability. A decrease in m on passing inwards would tend towards instability, but it hardly seems likely that m could vary rapidly enough (except near the surface) to produce instability.

Thus although our treatment has been in some ways unsatisfactory and incomplete, it seems fairly certain that the series of homologous configurations will be stable without convection currents being set up, except near the surface, an exception possibly arising when γ is very close to the value $\frac{4}{3}$.

13. The emission of radiation is given by equation (26). Using the approximate relation $\mu c = \frac{5}{6}$, this becomes

$$\mathrm{E} = \frac{10\pi\sigma\mathrm{M}}{3c}\frac{m\mathrm{T}^3}{\rho} \quad . \quad . \quad . \quad . \quad (29)$$

Here $m\mathrm{T}^3/\rho$ is the constant value of this quantity throughout the outer layer (*cf.* equation (24)), and E, as we have already seen, remains constant subject to the assumption that c may be regarded as constant.

This last assumption must be very far from the truth, for as a star gets hotter we are dealing with light of ever shorter wave-lengths, and the opacity c will decrease accordingly. An approximation to actual conditions will perhaps be obtained by assuming a gradual secular decrease of c, and this will involve a corresponding increase in E. Unfortunately our general ignorance of gaseous opacity puts any exact treatment out of the question.

A rough numerical estimate of the right-hand member of

equation (29) suggests that the emission is at least of the same order of magnitude as that of giant stars.

Summary and Discussion of Results.

14. We have considered the sequence of events in a gas radiating heat from its surface and endowed with no source of energy except its own gravitation. Masses of gas for which $\gamma > \frac{4}{3}$ will contract and become hotter as they lose heat; masses of gas for which $\gamma < \frac{4}{3}$ would expand and become cooler except that, being necessarily unstable, they cannot exist except in a state of infinite diffusion.

In discussing the contraction of a mass of gas, most previous investigators have been content to assume that the mass remains permanently in adiabatic equilibrium. We have, however, found that the adiabatic configuration does not possess the quality of permanence; a mass of gas started originally in an adiabatic state will depart from this state as contraction proceeds.

The question of whether any other type of configuration possesses the quality of permanence becomes of importance, and we have found that the answer depends on the nature of the mechanism by which the transfer of heat takes place. If transfer takes place purely by conduction, we have found that there is a permanent series of homologous configurations—*i.e.* a series such that any configuration can be derived from any other by uniform contraction or expansion,—this series being such that a contracting mass would pass, by natural shrinkage, through the different members of the series in turn. The same result is true when the transfer of heat takes place purely by radiation, if we assume the opacity c to have a constant value throughout. On the other hand, if transfer of heat takes place both by conduction and also by radiation, no such permanent series exists; in such a case the arrangement of the gas will change both proportionately and absolutely as heat is radiated away.

Under the conditions prevailing in giant stars we have seen (in the preceding paper) that probably far more heat is transferred by radiation than by conduction. Thus as an approximation we may suppose that all heat is transferred by radiation.

A permanent homologous series will now exist having the property that a configuration once on it will remain on it throughout the process of contraction. Although we have not been able to obtain a rigorous proof, yet, from analogy with other physical problems of a similar nature, there can be but little doubt that a configuration which is not initially on this series will approach asymptotically to it as contraction proceeds. Thus, as an approximation, we may suppose that all naturally contracting masses of gas have configurations on this series.

We have examined the mechanical stability of configurations on this series. A first condition for stability is that γ shall be greater than $\frac{4}{3}$. Confining our attention to masses of gas for which

γ has values greater than this, we have found (although by something short of strict proof) that the conditions of mechanical stability are likely to be satisfied except near the surface, a possible exception occurring when γ is only just greater than $\frac{4}{3}$. Near the surface stability may fail if there is a layer of high ionisation sufficiently near to the surface; if this occurs, a steady flow of convection currents will be set up in the layers which would otherwise be unstable, and stability will in this way be established.

Still assuming the opacity c to retain a constant value, we have found the total emission of radiation to remain unchanged as a star moves along this series.

The mechanism of this emission of radiation is of some interest. As a star contracts, the photosphere approaches the surface both relatively and absolutely. One consequence is that the actual emission of radiation comes from lighter and lighter elements; another is that the temperature T_1 of the photosphere increases less rapidly than the mean temperature of the gas. If $\bar{\rho}$ is the mean density of the gas, Lane's law states vaguely that the temperature varies as $(\bar{\rho})^{\frac{1}{3}}$. But if T_1 is the temperature of the photosphere and E the total emission, we have $E = 4\pi\sigma a^2 T_1^4$. On combining this with the relation $M = \frac{4}{3}\pi\bar{\rho}a^3$, we readily find that T_1 must actually vary as $E^{\frac{1}{4}}\bar{\rho}^{\frac{1}{12}}$. If E is supposed to remain constant, T_1 will vary as $\bar{\rho}^{\frac{1}{12}}$, a law which Eddington has already enunciated and shown to be in good agreement with estimates of the temperatures of stellar photospheres.

These simple laws are based on the assumption that the opacity retains a constant value; actually the opacity will decrease as the star gets hotter. This makes a permanent series of homologous configurations an impossibility, but we can perhaps get some idea of the actual motion by imagining that at any instant the mass is moving on the series appropriate to the value of c at that instant, while at the same time c experiences a secular decrease. This would result in E experiencing a secular increase. Thus in a cluster of stars of all ages the blue stars ought to be the brightest. This is the reverse of the law found by Shapley for the Hercules cluster, but this cluster is probably not a mixture of stars of all ages; if the stars were all of the same age, the giant red stars, being the least developed and therefore the most massive, might reasonably be expected to be the brightest.

Clearly we are led to the conception of visible giant stars being only the end of a long series of configurations of which the earlier members, being at much lower temperatures, will radiate only heat. The total emission of energy of these pre-giant stars will be very much less than that of the giant stars, so that a star may spend only a small fraction of its life as a visible giant star, and in a cluster of stars of all ages mixed a large proportion of the stars will be dark. This does something towards extending the life permitted to a star by the contraction-theory, but numerical data are so uncertain that it is hardly possible to conjecture what this length of life may be.

A Note on the Cusped Orbit of Dr. Hill.

By R. Moritz.

(*Communicated by P. H. Cowell, D.Sc., F.R.S.*)

Dr. Hill has pointed out in his classical paper "Researches in the Lunar Theory" * that, for satellites whose mean motions bear a ratio to that of their primary less than three, the method of solution of the equations of motion by trigonometrical series proceeding by multiples of the argument of the variation fails, and that recourse must be had to mechanical quadratures. He proceeded to develop formulæ suitable for this purpose, and applied them to the plotting of the orbit, which he erroneously described as that of maximum lunation, *i.e.* the orbit, cusped with reference to his system of moving axes, in which the velocity of the satellite vanishes in quadratures.

These formulæ are lengthy and inconvenient; an attempt to reproduce Dr. Hill's arithmetical result has convinced me that the labour of obtaining it must have been extremely heavy. Further, the formulæ themselves are not free from error, the coefficient of t^9 in the value of x at p. 331, *loc. cit.*, being incorrect, a circumstance which vitiates other parts of the same expressions for x and y. Nevertheless, as will appear in the sequel, these errors hardly seem heavy enough to affect the accuracy of the actual result published.

Dr. Cowell has pointed out to me that the method of mechanical quadratures, which he used with such success in following out the orbits of Halley's Comet and J VIII, is equally suitable for the investigation of this class of satellite. As an example, at his suggestion I have worked out the case of the cusped orbit above alluded to, and with his permission I append the details in order to demonstrate the simplicity of the method.

If one sought to compute this orbit *ab initio*, the method would be one of trial and error. Starting with an assumed value of the radius vector from quadrature, one would follow the orbit until this quantity became a minimum, and then ascertain whether at that moment the satellite were in syzygy. It would naturally be found not to be so exactly, and a further attempt would be made with another trial radius vector, taking care that upon this occasion the time of the radius vector becoming a minimum exhibited the satellite upon the opposite side of the line of syzygy to the first trial. From the two angular distances from syzygy thus obtained, an interpolation would provide an initial radius vector so near the true value that probably one more trial would suffice for a completely accurate result. As, however, Dr. Hill has published a value for the radius vector in quadrature appropriate to the case of the orbit in question, it will be simpler to use this value as an example of the method, simply seeking for the time of minimum

* *Collected Mathematical Works*, vol. i. p. 284.

radius vector, and then proceeding to inquire whether, as Dr. Hill states, at that time the satellite and the sun are in syzygy.

The method adopts fixed axes, and the units must be chosen so that the expressions for the forces are as simple as possible. Dr. Hill employed a system wherein both μ (the joint mass of earth and moon) and n' (the sun's motion) were unity. I will use a system wherein $\mu = 1$ but $n' = 3°$. As Dr. Hill's value of his radius vector in quadrature is ·78190 in his units, it follows that my value for the same quantity will be 5·5867.

The force function of the sun becomes

$$\frac{n'^2}{2}[2(x\cos n't + y\sin n't)^2 - (x\sin n't - y\cos n't)^2],$$

the forces due to the sun along the axes of x and y being respectively

$$\frac{n'^2}{2}[x(3\cos 2n't + 1) + 3y\sin 2n't] \text{ and } \frac{n'^2}{2}[3x\sin 2n't + y(-3\cos 2n't + 1)],$$

while the forces due to the primary along the same axes are $-\frac{x}{r^3}$ and $-\frac{y}{r^3}$. It is convenient first of all to compute for all values of odd multiples of $n't = 1°·5$ the value of the three functions (or their logarithms)

$$A = \frac{n'^2}{2}(3\cos 2n't + 1)$$

$$B = \frac{3n'^2}{2}\sin 2n't$$

$$C = \frac{n'^2}{2}(-3\cos 2n't + 1),$$

a process which does not occupy many minutes.

The forces along the axes are then

$$-\frac{x}{r^3} + Ax + By$$

and

$$-\frac{y}{r^3} + Bx + Cy \text{ respectively,}$$

and can be computed at each stage of the quadratures with great ease and rapidity. In what follows small letters denote the coordinates and capitals denote forces, suffixes indicating the differences of these quantities. Thus X is the force along the x axis, X_1 and X_2 being its first and second differences, while X_{-1} and X_{-2} are the first and second built-up "back-differences," if I may coin the expression.

I may briefly recall that the process depends on the formula for double integration

$$x = X_{-2} + \tfrac{1}{12}X - \tfrac{1}{240}X_2 + \tfrac{31}{60480}X_4 - \ldots,$$

the principle being at each stage of the quadratures to select a tentative value for x and y, form the forces X and Y, thence the forward and backward differences X_1, X_2, X_{-1}, X_{-2}, etc., and thence by the above formula obtain the integrated values of x and y. If these do not agree within a few units in the last place with the tentative values from which the forces were computed, the process must be repeated with the new values. The method is therefore quite mechanical, save for the choice at each stage of these tentative values of the co-ordinates, which is to say, in other words, judging from previous values of $\frac{1}{12}X$ and $\frac{1}{12}Y$, their probable values for the new stage. If the intervals are chosen sufficiently small, this may be done so accurately by inspection of the differences X_1 and Y_1 that no repetition is necessary at any stage.

Below are the details of the computations. All three sheets are used in turn. The forces are first worked on the tentative sheet, and then the accurate concluded values of the co-ordinates are obtained on the x and y sheets. Thence new tentative values are inferred, and a fresh stage is begun on the first sheet in a way which will be obvious.

$t+\frac{1}{2}$.	Tentative. x.	Tentative. y.	Tentative. r^2.	$-\frac{x}{r^3}$.	Ax.	By.	$-\frac{y}{r^3}$.	Bx.	Cy.
0	−0·1462	−5·5808	...	...	...	...	...	...	...
1	+0·1462	−5·5808	31·1667	− 8	+ 8	− 12	+ 321	0	+153
2	0·4374	−5·5333	30·8087	− 26	+24	− 36	+ 324	+ 3	+149
3	0·7248	−5·4383	30·1004	− 44	+39	− 58	+ 329	+ 8	+141
4	1·0058	−5·2955	29·0539	− 64	+52	− 78	+ 338	+ 15	+131
5	1·2779	−5·1042	27·6859	− 88	+64	− 95	+ 350	+ 24	+117
6	1·5380	−4·8638	26·0220	− 116	+74	−109	+ 366	+ 34	+101
7	1·7830	−4·5733	24·0942	− 151	+81	−118	+ 387	+ 46	+ 83
8	2·0092	−4·2311	21·9391	− 196	+86	−123	+ 412	+ 58	+ 65
9	2·2119	−3·8354	19·6028	− 255	+88	−123	+ 442	+ 71	+ 47
10	2·3854	−3·3837	17·1395	− 336	+86	−117	+ 477	+ 82	+ 29
11	2·5219	−2·8726	14·6118	− 452	+82	−105	+ 514	+ 92	+ 14
12	2·6106	−2·3006	12·1080	− 620	+74	− 88	+ 546	+100	+ 2
13	2·6353	−1·6635	9·7120	− 871	+64	− 66	+ 550	+105	− 5
14	2·5717	−0·9620	7·5391	−1242	+52	− 39	+ 465	+104	− 7
15	2·3840	−0·2063	5·7261	−1740	+38	− 8	+ 151	+ 98	− 2
16	2·0259	+0·5702	4·4294	−2173	+23	+ 23	− 611	+ 83	+ 9
17	1·4596	+1·2920	3·7997	−1971	+11	+ 52	−1744	+ 59	+ 26
18	+0·7110	+1·8514	3·9332	− 911	+ 2	+ 74	−2374	+ 28	+ 45
19	−0·1216	+2·1887	4·8052	+ 115	0	+ 84	−2078	− 5	+ 62

$t+\frac{1}{2}$.	X_{-2}.	X_{-1}.	X.	X_1.	X_2.	$\frac{1}{12}X$.	$-\frac{1}{240}X_2$.	Concluded x.
0	−0·1463	...	...	...	...	...	...	−0·1462
1	+0·1463	+2926	− 12	...	...	− 1	...	+0·1462
2	+0·4377	+2914	− 38	− 26	+ 1	− 3	...	+0·4374
3	+0·7253	+2876	− 63	− 25	− 2	− 5	...	+0·7248
4	+1·0066	+2813	− 90	− 27	− 2	− 8	...	+1·0058
5	+1·2789	+2723	− 119	− 29	− 3	− 10	...	+1·2779
6	+1·5393	+2604	− 151	− 32	− 5	− 13	...	+1·5380
7	+1·7846	+2453	− 188	− 37	− 8	− 16	...	+1·7830
8	+2·0111	+2265	− 233	− 45	− 12	− 19	...	+2·0092
9	+2·2143	+2032	− 290	− 57	− 20	− 24	...	+2·2119
10	+2·3885	+1742	− 367	− 77	− 31	− 31	...	+2·3854
11	+2·5260	+1375	− 475	− 108	− 51	− 40	...	+2·5220
12	+2·6160	+ 900	− 634	− 159	− 80	− 53	...	+2·6107
13	+2·6426	+ 266	− 873	− 239	−117	− 73	...	+2·6353
14	+2·5819	− 607	−1229	− 356	−125	−102	+1	+2·5717
15	+2·3983	−1836	−1710	− 481	+ 64	−143	...	+2·3841
16	+2·0437	−3546	−2127	− 417	+638	−177	− 3	+2·0262
17	+1·4764	−5673	−1908	+ 221	+852	−167	− 4	+1·4594
18	+0·7183	−7581	− 835	+1073	− 39	− 70	...	+0·7109
19	−0·1233	−8416	+ 199	+1034	...	+ 17	...	−0·1216

$t+\frac{1}{2}$.	Y_{-2}.	Y_{-1}.	Y.	Y_1.	Y_2.	$\frac{1}{12}Y$.	$-\frac{1}{240}Y_2$.	Concluded y.
0	−5·5847		...		...	+ 40	...	−5·5807
1	−5·5847	0	+ 474	...	...	+ 40	...	−5·5807
2	−5·5373	+ 474	+ 476	+ 2	0	+ 40	...	−5·5333
3	−5·4423	+ 950	+ 478	+ 2	+ 4	+ 40	...	−5·4383
4	−5·2995	+1428	+ 484	+ 6	+ 1	+ 40	...	−5·2955
5	−5·1083	+1912	+ 491	+ 7	+ 3	+ 41	...	−5·1042
6	−4·8680	+2403	+ 501	+ 10	+ 5	+ 42	...	−4·8638
7	−4·5776	+2904	+ 516	+ 15	+ 4	+ 43	...	−4·5733
8	−4·2356	+3420	+ 535	+ 19	+ 6	+ 45	...	−4·2311
9	−3·8401	+3955	+ 560	+ 25	+ 3	+ 47	...	−3·8354
10	−3·3886	+4515	+ 588	+ 28	+ 4	+ 49	...	−3·3837
11	−2·8783	+5103	+ 620	+ 32	− 4	+ 52	...	−2·8731
12	−2·3060	+5723	+ 648	+ 28	− 26	+ 54	...	−2·3006
13	−1·6689	+6371	+ 650	+ 2	− 90	+ 54	...	−1·6635
14	−0·9668	+7021	+ 562	− 88	−227	+ 47	+1	−0·9621
15	−0·2085	+7583	+ 247	− 315	−451	+ 21	+2	−0·2063
16	+0·5745	+7830	− 519	− 766	−374	− 73	+2	+0·5704
17	+1·3056	+7311	−1659	−1140	+498	−138	− 2	+1·2920
18	+1·8708	+5652	−2301	− 642	+922	−192	− 4	+1·8514
19	+2·2059	+3351	−2021	+ 280	...	−168	...	+2·1887

It may not be superfluous to explain in detail how the quantities which head these tables are obtained. It is required to calculate an orbit in which at time $t = 0$

$$y = -5{\cdot}5867$$
$$\frac{d\theta}{dt} = 3^\circ$$

or $\frac{dx}{dt} = +5{\cdot}5867 \times$ circular measure of $3^\circ = +0{\cdot}2925$.

The formulæ to be used are, like the fundamental formula, due to Gauss, and are:

$$(y)_{t=0} = \overline{Y}_{-2} - \tfrac{1}{24}\overline{Y} \quad . \quad . \quad . \quad (a)$$

and

$$\left(\frac{dx}{dt}\right)_{t=0} = X_{-1} + \tfrac{1}{24}X_1 \quad . \quad . \quad . \quad (b)$$

where in (a) the bars denote the *mean* of the values for $t = \pm 0{\cdot}5$. By symmetry x, X change sign with the time, while y, Y are unaltered. Hence from (a)

$$(Y_{-2})_{t=0{\cdot}5} = -5{\cdot}5867 + \tfrac{1}{24}Y \quad . \quad . \quad . \quad (c)$$

and from (b)

$$(X_{-1})_{t=0} = 2(X_{-2})_{t=0{\cdot}5} = 0{\cdot}2925 - \tfrac{1}{24}X_1$$

or

$$2X_{-2} = 0{\cdot}2925 - \tfrac{1}{12}X \text{ at time } t = 0{\cdot}5 \quad . \quad (d)$$

To get values of $\frac{1}{24}Y$ and $\frac{1}{12}X$ for use in formulæ (c) and (d), the roughest of trial values of the co-ordinates suffice. Assuming therefore that the moon remains in quadrature and performs a circular path from time $t = 0$ to $t = 0{\cdot}5$, which is not far from the truth,

$$x = 5{\cdot}5867 \sin 1^\circ{\cdot}5 = 0{\cdot}1462,$$
$$y = -5{\cdot}5867 \cos 1^\circ{\cdot}5 = -5{\cdot}5847,$$

whence

$$\left.\begin{aligned} X &= -0{\cdot}0012 \\ Y &= +0{\cdot}0474 \end{aligned}\right\} \text{ for time } t = 0{\cdot}5,$$

whence

$$X_{-2} = +0{\cdot}1463,$$
$$Y_{-2} = -5{\cdot}5847.$$

Now, using the fundamental formula $x = X_{-2} + \frac{1}{12}X$, we get

$$\left.\begin{aligned} x &= \pm 0{\cdot}1462 \\ y &= -5{\cdot}5808 \end{aligned}\right\} \text{ for time } t = \pm 0{\cdot}5.$$

These are accurate values satisfying the initial conditions. We now recalculate the forces with these co-ordinates and proceed with the quadratures.

It will be observed that the value of r^2 passes a minimum between lines 16 and 18. Line 19 is merely computed for the purpose of obtaining a fifth difference at line 16·5. Having differenced these values, we find by interpolation to fifth differences that r^2 is a minimum at time $t = 16{\cdot}8289$. Now, as Dr. Hill states that his cusped orbit has 1·78265 lunations in a year, it follows that the sun moves through 50°·4866, while the argument of the variation moves through one right angle. In time $t = 16{\cdot}8289$ the sun has moved through exactly 50°·4867. It is therefore only necessary to inquire what angle the satellite has moved through in this time. Interpolating the angle $\tan^{-1}\frac{y}{x}$ from the above values of the co-ordinates at time $t = 16{\cdot}8289$, it is found that the moon has moved through

$$90^\circ + 50^\circ\ 36'\ 3'',$$

while the sun has moved through 50° 29′ 12″, pointing to syzygy within 7′ of arc.

This is quite as close accordance as is to be expected, having regard to the fairly large interval chosen and to the fact that the angular velocity of the moon is very large in quadrature. Dr. Hill's orbit is therefore confirmed.

The whole of these quadratures can easily be performed in three hours by an experienced computer, from which it will be seen how favourably the method compares with that of Dr. Hill, by anyone who has experience of the latter.

A table of the values of the co-ordinates referred to moving axes is appended. They serve to plot the orbit, which, however, I refrain from drawing, as Dr. Hill has given it in the paper referred to. At time $t = 0$, $x = 0$, and $y = 5{\cdot}5867$.

t.	*x.*	*y.*	*t.*	*x.*	*y.*	*t.*	*x.*	*y.*
0·5	0·0000	+5·5827	6·5	0·1541	+4·9063	12·5	1·0780	+2·9240
1·5	0·0029	5·5507	7·5	0·2371	4·6779	13·5	1·3307	2·3992
2·5	0·0088	5·4865	8·5	0·3452	4·3136	14·5	1·5874	1·7907
3·5	0·0247	5·3902	9·5	0·4818	4·1119	15·5	1·8085	1·0771
4·5	0·0510	5·2618	10·5	0·6491	3·7674	16·5	1·9303	+0·2707
5·5	0·0933	+5·1004	11·5	0·8486	+3·3747	17·5	1·9016	−0·5631

I should add that the method could be applied at once to those looped orbits which M. Poincaré showed to succeed the case of the cusped orbit, when the ratio of the mean motions is still further reduced.

Highgate:
1917 *Sept.*

The Stellar Magnitude Scales of the Astrographic Catalogue. Eleventh Note: The Cordoba Magnitudes ($-25°$) *and the Tacubaya Magnitudes* ($-15°$). *With a further Note on the Progression of the Galactic Condensation.* By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The primary object of these notes is, as indicated in the general heading, to furnish material for the intercomparison of the magnitude scales of the *Astrographic Catalogue* on the plan outlined in 1909 March. Provisional information has already been given for the Vatican, Oxford, Paris, Bordeaux, Toulouse, Algiers, Perth, Cape and Melbourne zones: to which list may be added Hyderabad and Edinburgh (=Perth), supplied by the directors of those observatories: and it is now a pleasure to be able to add zones from Cordoba and Tacubaya, counts of which have been kindly supplied in advance of publication. Tacubaya promises to add another zone ($-16°$) shortly: some hours of it have already been received. A good deal of new material relating to other observatories, of which at least one zone has already been given, is also in hand, and will be given in due course.

2. As regards the remaining observatories—

At Greenwich thorough investigations are being made on independent lines.

Catania has adopted the plan of reducing and publishing the first six hours of each zone, in preference to completing any one zone. This renders it impossible to make an investigation on the adopted lines.

Helsingfors has also adopted a plan of this kind.

Potsdam has published in miscellaneous order. If the distribution had been equable, it would have been possible to extract the equivalent of a zone: but there are very few plates near $22-23^h$ in any zone.

San Fernando: nothing published, so far as I am aware.

Santiago: measurement proceeding, but nothing yet published.

Sydney: nothing yet published, so far as I am aware.

3. In 1909 the evidence of some early counts seemed to discredit Kapteyn's results for the progression of the galactic condensation: or rather to show that this progression might not be the same for photographic magnitudes as Kapteyn had obtained for visual magnitudes. This was gradually found to be a spurious result: and more than two years ago * (*M.N.*, 1915 April, p. 481; 1915 June, p. 608) a value of the progression not far from Kapteyn's was provisionally inferred from the material then collected.

The earlier spurious result was due to two causes:—

Firstly, the use of limited material which was not thoroughly representative of the sky. With some diffidence it is submitted

* But see also Mr. Pocock's paper in 1915 Jan., p. 149. This paper was written by him more than a year previously, as stated in an introductory note.

that the same cause led Chapman and Melotte also to a too small value of the progression.

Secondly, the existence of another definite phenomenon in the sky, provisionally described as a spiral of obscuration.

4. In a recent paper (*Ap. J.*, 1917 Sept.), Mr. F. H. Seares has challenged the existence of this second phenomenon. A reply on that point has been sent to the same journal: but since the decision will ultimately lie with the facts which are being steadily accumulated in these papers, only brief mention of his points is made here. In the same paper, however, Seares has given a valuable determination of the progression of the galactic condensation, which may be used as a good first approximation in further work. Since the existence of this progression was definitely realised in 1915, it has always been the intention ultimately to clear the counts from its effects for a definitive determination of the "spiral": but it was natural to defer the discussion during the accumulation of material. Seares's results, however, agree so closely with Kapteyn's that it seems needless to defer longer the adoption of a first approximation.

5. Moreover, it is satisfactory to find that they agree closely in essentials with the result provisionally given in 1915 June, as follows (*M.N.*, lxxv. p. 608):—

The general mean [value of the change of log N per magnitude, per degree of galactic latitude] seems to be about $-\cdot0010$ to $-\cdot0015$, and applies to the neighbourhood of magnitude 11·5 (scale of Chapman and Melotte). Kapteyn's visual mean coefficient in this neighbourhood is about $-\cdot0016$, which is of the same order though rather larger.

6. Let us accept this statement as a rigid one, though it was clearly not so intended. We shall find that it guides us close to Seares's results. We must replace the limits $-\cdot0010$ to $-\cdot0015$ by the mid-value $-\cdot00125$. But I propose to get rid of the magnitude scale altogether for the present, and substitute that of log N, about which there is no doubt. About the magnitude scale there are considerable doubts, as Seares* and Sampson† have both shown. And we do not really want it at present. What we want to know really is this: given $\log N_\gamma$ for any galactic latitude γ, what is the corresponding value $\log N_0$ on the galactic equator?

7. Now the statement above quoted implies a formula

$$\log N_0 = \log N_\gamma(1 + k\gamma) + K_\gamma,$$

and the coefficient k is the rate of change of log N per degree of galactic latitude *not* for one magnitude, but for one unit in log N, which corresponds to about two magnitudes. Hence $k = \cdot00125 \times 2 = \cdot0025$, which will be adopted for use.

8. Now consider Seares's table iii. (*loc. cit.*) ($\log N_m$ = number of stars per square degree brighter than m).

* *Proc. Nat. Acad. Sci. U.S.A.*, vol. iii. (1917), p. 217.
† *Monthly Notices*, lxxvii. p. 613.

m \ γ	0°.	10°.	20°.	30°.	40°.	50°.	60°.	70°.	80°.
8·5	0·50	0·45	0·32	0·24	0·18	0·14	0·12	0·11	0·10
9·0	0·75	0·70	0·56	0·46	0·39	0·34	0·31	0·31	0·30
9·5	1·03	0·96	0·82	0·70	0·61	0·55	0·52	0·51	0·50
10·0	1·29	1·20	1·06	0·94	0·83	0·77	0·73	0·72	0·72
10·5	1·52	1·44	1·30	1·16	1·04	0·98	0·93	0·92	0·92
11·0	1·75	1·67	1·53	1·38	1·25	1·19	1·14	1·12	1·11
11·5	1·99	1·90	1·75	1·60	1·47	1·40	1·35	1·32	1·29
12·0	2·22	2·13	1·97	1·82	1·68	1·60	1·55	1·51	1·46
12·5	2·45	2·36	2·19	2·03	1·90	1·81	1·74	1·69	1·63

To test the above formula we have only to form for each column the quantity

$$\log N_0 - \log N_\gamma(1 + k\gamma),$$

which should be equal to K_γ, and therefore constant for each column. The values of K_γ are given below in Table I.

TABLE I.

Values of K_γ deduced from Seares's Table III. in Units of 0·01.

log N_0.	10°.	20°.	30°.	40°.	50°.	60°.	70°.	80°.
0·50	4	16	24	30	34	36	37	38
0·75	3	16	26	32	37	39	39	39
1·03	5	17	28	36	41	43	43	43
1·29	6	18	28	38	42	45	44	43
1·52	4	16	27	38	42	45	44	42
1·75	4	14	27	38	41	44	43	42
1·99	4	15	27	37	41	44	44	44
2·22	4	15	26	37	42	44	45	47
2·45	3	15	27	36	41	45	46	49
Mean	4	16	27	37	41	44	44	44

9. The brightest stars stand out a little, especially in high latitudes; but in the main body of the table the deviations from the *adopted* means at the foot rarely exceed one unit, *i.e.* 0·01. To prevent misunderstanding it is here repeated that so close an accordance with the simple linear formula came as a surprise: the result sketched in 1915 was not intended to be taken so literally. If we are to adopt a single factor k for the product $\gamma \times \log N$, we can clearly not improve on the adopted ·0025 without upsetting the body of the table to suit the few brightest stars. It seems better to reserve a special correction for them, and to adopt the very simple formula

$$\log N_0 = (1 + \cdot0025\gamma)\log N_\gamma + K_\gamma,$$

where γ is the galactic latitude in degrees and K_γ depends on γ only.

10. The value of K_γ will, however be altered if we adopt as the unit area the astrographic plate of about 4 square degrees instead of the square degree: for when we add $\log 4 = 0{\cdot}60$ to $\log N_\gamma$, we get the extra term $+{\cdot}0015\gamma$, which rises to ·135 when $\gamma = 90^\circ$. There are some advantages for this particular work in retaining the astrographic plate as unit, but on consideration I do not think they are great enough to set against uniformity of practice generally; and as a revised reduction is being made, we will adopt the square degree as unit throughout the work from this point.

11. It remains to tabulate K_γ more in detail, as in Table II.

TABLE II.

Adopted Values of K_γ *in the formula* $\log N_0 = (1 + {\cdot}0025\gamma)\log N_\gamma + K_\gamma$ *in Units of* 0·01 *: N referring to one square degree.*

c	0°.	1°.	2°.	3°.	4°.	5°.	6°.	7°.	8°.	9°.
0	0	0	0	1	1	1	2	2	3	3
10	4	5	6	7	9	10	11	12	13	15
20	16	17	19	20	21	22	23	24	25	26
30	27	29	30	31	32	33	34	35	35	36
40	37	37	38	38	39	39	40	40	40	41
50	41	41	41	42	42	42	43	43	43	44
60	44	44	44	44	44	44	44	44	44	44
70	44	44	44	44	44	44	44	44	44	44
80	44	44	44	44	44	44	44	44	44	44

To obtain complete agreement with Seares's table iii. a correction to K_γ is required of $-0{\cdot}07$ for $\log N_0 = 0{\cdot}50$, and $-0{\cdot}05$ for $\log N_0 = 0{\cdot}75$; constant for all latitudes exceeding $\gamma = 35^\circ$ say.

12. A word may now be said about Seares's suggestion that the "spiral" is a spurious effect of omitting to clear the results of the galactic progression. Undoubtedly they should be so cleared either first or last. Thanks to Seares's investigation, there is now no difficulty in clearing them as we go along, and this will now be done, those previously given being also duly cleared as opportunity offers. As hitherto given the spiral is a composite effect,

spiral = galactic progression + something else,

and the something else will obviously differ in important respects from the spiral. But it can scarcely be a negligible phenomenon, leaving the two others as equivalent, for they are completely different in character. The Galaxy is symmetrical about the plane through its poles and the earth's poles; hence any effect of galactic latitude can have maxima or minima in that plane alone. If we formed the first harmonic for any zone, say $k\cos(\theta - \theta_0)$, then θ_0 could only have the value 10° or 190°. Now the characteristic of the "spiral" is the rapid variation of θ_0 with δ. It is difficult to see how one phenomenon could be mistaken for the other.

13. We now proceed to treat the material kindly furnished by the Cordoba and Tacubaya observatories in the light of these

results. In Table III. are given the figures supplied by Dr. Perrine for zone $-25°$, except that for diameters over 70 the detailed counts given by him have been summarised in one column. The next columns give the number of diameters between 61 and 70 inclusive, between 51 and 60, and so on, with totals at the end. He adds, however, a note:—"The diameters of 58 stars have not been measured, being too faint or too large."

The counts are thus liable to an unknown disturbance from these 58 stars, which may be serious in the earlier columns. It seems therefore scarcely advisable to deal seriously with the counts before the column 51–60. Forming from this point the cumulative counts N′, dividing by 7 or 8 according to number of plates shown in second column, and again by 4 on the assumption that the area of each plate may be taken as 4 square degrees, we then form log N_γ. We then reduce this to the Galaxy by the above formula and Table II.: so that if the galactic phenomena were the only

TABLE III.

Cordoba Zone $-25°$.

Hour.	No. Plates.	Over 70.	61–70.	51–60.	41–50.	31–40.	21–30.	11–20.	1–10.	Total.
0	7	3	0	4	12	37	146	489	254	945
1	8	4	1	4	11	77	201	769	315	1382
2	7	2	0	4	13	35	160	667	471	1352
3	8	2	3	5	29	52	223	794	391	1499
4	7	0	2	7	21	50	209	904	685	1878
5	8	2	1	7	40	137	470	1631	797	3085
6	7	1	3	15	67	191	742	2342	895	4256
7	8	7	6	18	56	254	1420	5063	3694	10518
8	7	3	7	7	46	243	864	3572	1329	6071
9	8	3	4	12	30	139	492	1691	879	3250
10	7	1	1	4	23	92	305	870	1163	2459
11	8	1	4	5	31	121	352	962	464	1940
12	7	0	1	4	13	58	230	1083	638	2027
13	8	2	1	7	24	122	300	1088	433	1977
14	7	3	1	3	13	39	236	841	428	1564
15	8	1	4	5	22	69	224	1007	660	1992
16	7	1	4	5	26	70	306	1255	747	2414
17	8	5	1	6	30	203	580	1745	1236	3806
18	7	4	1	8	40	157	623	2376	1426	4635
19	8	2	4	5	32	107	522	2375	1282	4329
20	7	0	3	8	17	92	137	1672	850	2779
21	8	1	2	3	27	74	252	967	745	2071
22	7	2	2	1	12	36	144	550	544	1291
23	8	3	2	5	14	39	170	668	343	1244
Totals		53	58	152	649	2494	9308	35381	20669	68764
N′		53	111	263	912	3406	12714	48095	68764	
Cordoba Mag.		5·8	6·5	7·1	7·8	8·6	9·2	9·9	10·6	
C. & M. Mag.		6·1	6·7	7·4	8·5	9·7	11·1	12·6	13·1	

ones, we should get a series of uniform values of log N_0. The mean of these 24 values is adopted as the index of the particular magnitude under investigation. Its place in a standard scale of magnitudes is reserved for future consideration.

14. This index is now placed at the head of the column as log N_0, and the differences for individual hours are shown in Table IV. The second column shows the galactic latitude for the middle of the hour of R.A., the third the factor for multiplying log N_γ. The derivation of this factor from the galactic latitude is so simple that it is perhaps superfluous to give it; but in this first example it is better to be clear. The fourth column contains the value of K_γ, which is constant for all the columns, and does not therefore affect any discussion of *differences* such as has been previously given: these are only affected by the products of log N_γ by the factors.

Then follow the residuals for the various columns, in units of 0·01 to avoid decimals. The decimals are, however, retained in the headings, which are the values of log N_0.

TABLE IV.

Cordoba −25°.

R.A.	γ.	Factor.	Kγ.	9·73.	0·32.	0·92.	1·52.	2·15.	2·33.	(α.)	(β.)
0	−87°	·218	·44	− 2	− 9	−11	− 3	− 2	− 3	+ 5	−13
1	−79	·198	·44	+ 5	−12	+11	+ 8	+12	+ 7	+10	− 7
2	−66	·165	·44	− 7	− 8	− 5	− 5	+ 4	+ 7	+12	0
3	−53	·132	·42	+12	+20	+ 2	+ 3	+ 1	− 1	−10	−15
4	−40	·100	·37	+ 9	+ 8	− 5	− 4	+ 1	+ 4	− 4	− 5
5	−26	·065	·23	− 3	+12	+13	+12	+ 6	+ 2	− 1	+ 2
6	−14	·035	·09	+18	+27	+19	+17	+ 9	+ 2	−12	− 4
7	− 2	·005	·00	+26	+12	+12	+24	+20	+20	+ 5	+17
8	+10	·025	·04	+ 8	+ 8	+19	+18	+18	+10	+ 2	+13
9	+20	·050	·16	+20	+ 4	+ 5	+ 4	− 3	− 6	−11	− 5
10	+28	·070	·25	−20	− 6	+ 1	0	−11	+ 1	+ 5	+ 6
11	+35	·088	·33	+ 6	+13	+18	+13	0	− 5	−10	− 9
12	+37	·092	·35	−20	−18	−10	− 3	+ 5	+ 5	+19	+18
13	+36	·090	·34	+ 7	+ 5	+17	+ 8	+ 3	− 4	− 7	− 7
14	+31	·078	·29	− 9	−19	−29	−13	−14	−16	+ 5	+ 5
15	+24	·060	·21	− 5	−10	−18	−24	−23	−21	−11	− 7
16	+14	·035	·09	−11	−12	−23	−22	−23	−24	− 8	0
17	+ 3	·008	·01	−14	−19	− 1	− 8	−22	−22	− 6	+ 6
18	− 9	·022	·03	− 5	− 1	0	+ 1	− 1	− 3	+ 1	+11
19	−22	·055	·19	− 2	+ 1	− 2	+ 7	+13	+12	+12	+16
20	−35	·088	·33	+15	+ 1	+10	−15	+18	+17	− 2	− 2
21	−48	·120	·40	−14	+10	+ 7	+ 7	+ 6	+11	+ 7	+ 1
22	−61	·152	·44	−15	−13	−17	−11	− 7	+ 2	+11	− 1
23	−74	·185	·44	+ 2	− 2	−12	− 5	+ 1	− 1	+ 2	−13
m_p				7·5	8·2	8·9	9·6	10·3	11·0		
m_s					8·2	9·3	10·5	11·8	12·1		
m_k				7·0	8·1	9·3	10·5	11·8	12·2		
m_c				7·4	8·5	9·7	11·1	12·6	13·1		

15. At the foot of Table III. are given the magnitudes assigned by Perrine for the column. His note is "these magnitudes were obtained graphically from the curve passing through 61–70 and 21–30, the mean magnitudes of these groups being obtained from the Cordoba *D.M.*, vol. 16." It seems probable that his figures refer to the *mean* diameters of the columns, and not to the limiting diameters. In transferring them to Table IV. this assumption has been made, obtaining the values m_μ shown at the foot.

m_κ is taken from Seares's table iii., or Table I. of the present paper.

m_k from *Gron. Pub.*, 18.

m_c is from Chapman and Melotte's scale, as on p. 145 of *M.N.*, lxxv., deduced at foot of Table III. There is sufficient discrepancy to justify the policy of using simply log N_0 at present, without attempting to assign the magnitude scale.

16. We can now analyse the residuals of each column separately, without taking differences as before. Differences were formerly taken in order to eliminate the column K_γ, which is the same for all columns, but had not been precisely assigned.

By harmonic analysis the following expressions were obtained:—

TABLE V.

Cordoba $-25°$.

Harmonic Analysis of the Residuals of Table IV.

log N_0.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
9·73	+·082	+·005	−·012	−·036	−·004	+·007	+·050	−·014
0·32	+·096	+·003	−·033	−·058	−·053	−·032	−·010	+·008
0·92	+·082	−·026	−·068	−·044	+·008	−·038	+·059	+·024
1·52	+·105	−·018	−·060	−·012	−·017	−·011	+·057	+·026
2·15	+·077	+·043	−·068	−·005	−·020	−·039	+·096	−·010
2·33	+·066	+·054	−·083	+·008	−·011	−·043	+·063	−·030

17. In previous papers attention has been restricted practically to the first harmonic, and for the present we shall continue the restriction, merely tabulating the others for consideration later. It is doubtful whether the first two rows contain enough stars to be trustworthy. Subtracting the mean of the next two from the mean of the last two, we get for the series of coefficients—

$$-·024 \quad +·070 \quad -·014 \quad +·028 \quad -·007 \quad -·017 \quad +·022 \quad -·043,$$

corresponding to a difference of 1·02 in log N_0.

The first harmonic, reduced to a difference 1·00, is thus

$$-·074 \cos(\theta - 161°).$$

Now the value of θ_0 for the "spiral" is $247° + 92° = 339°$, which is undoubtedly quite different from 161°: but it was not to be

expected that when the galactic progression was subtracted the spiral would remain unchanged.

If we include the brighter stars, and subtract the mean of first three columns from mean of last three, we get for the coefficients (corresponding to 1·68 difference in log N)—

$$-\cdot005 \quad +\cdot020 \quad -\cdot034 \quad +\cdot049 \quad +\cdot003 \quad -\cdot010 \quad +\cdot039 \quad -\cdot008,$$

and for the first harmonic, reduced to difference 1·00, we get $-\cdot013 \cos(\theta - 166^\circ)$.

18. But this Cordoba zone would not in any case have offered very strong evidence in support of the "spiral." If we had formed the differences without first clearing the galactic condensation, we should have obtained, from last two pairs,

$$-\cdot026 \sin\theta + \cdot019 \cos\theta = -\cdot032 \cos(\theta - 127^\circ),$$

and from last three and first three, reducing to a difference of 1·00 in log N_0,

$$-\cdot011 \sin\theta - \cdot026 \cos\theta = -\cdot028 \cos(\theta - 23^\circ).$$

The coefficients are small and the phases not accordant.

19. But there is a definite "obscuration" effect, as can be seen by dividing the R.A. circuit into two portions, guided by the column (α), which shows the individual differences between means of first and last three. The mean values for the two portions are—

	9·73.	0·32.	0·92.	1·52.	2·15.	2·33.
$18^h - 2^h$	−3	−4	−2	−2	+5	+6
$3 - 17$	+2	+3	+2	+3	−4	−6

A glance at Table IV. will suffice to show that the division between these two portions is fairly sharp, which is fair evidence against a seasonal effect.

20. The column (β) shows the same differences before the galactic phenomena were eliminated. The appropriate division was then—

	9·73.	0·32.	0·92.	1·52.	2·15.	2·33.
$20^h - 6^h$	−6	−6	−11	−14	−11	−13
$7 - 19$	+4	+6	+10	+12	+13	+11

Tacubaya −15°.

21. Let us now turn to the other new contribution. Table VI. gives the counts as kindly supplied by the Director of the Tacubaya Observatory. He gave details for the bright stars, which it is not considered necessary to reproduce here. The magnitude scale for the bright stars will probably be discussed by other methods, and in any case their number is so few that the analysis is affected with large accidental errors.

The columns "Brt.," including all stars up to and including mag. 8·0 on the Tacubaya scale, with corresponding cumulative

sums for 9·0, 10·0, 11·0, and "All," have been selected for examination; and Table VII. is similar to Table IV. for Cordoba.

22. The columns of the table were then submitted to harmonic analysis, and the results are shown in Table VIII., which corresponds to Table V. for Cordoba. But there are certain broad features which can be expressed more readily by simple means as follows:—

R.A.	0·62.	1·22.	1·76.	2·16.	2·27.
$21^h - 5^h$	+13	+12	+ 8	+ 1	0
$6 - 10$	− 4	− 7	− 5	+ 6	+11
$11 - 17$	−14	−14	−10	−12	−15
$18 - 20$	− 4	+ 8	+ 8	+14	+18

The progression from $21^h - 5^h$ is in one direction, for $6^h - 10^h$ and $18^h - 20^h$ in the reverse, and for $11^h - 17^h$ sensibly zero.

TABLE VI.

Tacubaya Zone $-15°$.

Hour.	No. Plates.	Brt.	8·5.	9·0.	9·5.	10·0.	10·5.	11·0.	11·5.	Over.
0	7	54	50	89	128	145	193	132	185	
1	8	53	52	113	182	153	270	213	132	1
2	7	53	24	67	93	170	178	144	427	89
3	8	54	26	93	97	195	180	246	398	2
4	7	67	42	76	126	162	176	271	173	17
5	8	90	95	230	364	295	357	661	652	40
6	7	74	85	187	281	470	922	1764	1385	
7	8	96	88	194	421	1019	1755	2492	3328	
8	7	149	98	145	397	532	1244	1811	1502	
9	8	58	42	109	138	285	479	444	744	
10	7	45	28	69	100	181	379	489	586	
11	8	43	54	55	111	271	325	321	287	
12	7	61	32	50	57	189	217	240	277	
13	8	47	58	63	107	254	341	356	311	
14	7	40	42	58	101	276	486	334	176	
15	8	44	46	87	113	337	461	487	212	
16	7	39	37	57	163	252	436	69	44	
17	8	53	60	153	220	543	412	413	289	
18	7	77	161	212	347	829	1307	1881	1596	
19	8	83	59	219	300	545	845	1594	2034	
20	7	69	73	197	156	497	958	467	703	
21	8	92	74	112	142	352	491	460	444	
22	7	64	61	46	138	247	327	199	98	
23	8	64	53	94	84	367	638	273	108	
Totals		1569	1440	2775	4366	8566	13177	15761	16091	149
N′		1569	3009	5784	10150	18716	31893	47654	63745	63894

TABLE VII.

Tacubaya $-15°$.

R.A.	γ.	Factor.	K_γ.	0·62.	1·22.	1·76.	2·16.	2·27.
0^h	$-77°$	·192	·44	+16	+22	+14	+ 1	0
1	−73	·182	·44	+ 8	+21	+14	+ 7	+ 2
2	−62	·155	·44	+12	+ 3	+ 1	−10	+ 7
3	−49	·122	·41	+ 3	0	− 6	−14	− 7
4	−36	·090	·34	+12	0	−13	−19	−20
5	−23	·058	·20	+ 5	+16	+ 6	− 3	− 2
6	−10	·025	·15	−15	− 7	−10	+ 5	+ 8
7	+ 4	·010	·01	−12	−11	+ 3	+16	+24
8	+16	·040	·11	+24	+ 8	+ 9	+23	+26
9	+27	·068	·24	−10	−10	−13	−11	− 4
10	+36	·090	·34	− 8	−13	−15	− 3	+ 3
11	+44	·110	·39	−10	− 9	− 2	− 4	− 4
12	+46	·115	·40	+14	− 4	−10	−12	−10
13	+46	·115	·40	− 5	− 2	− 1	0	0
14	+40	·100	·37	− 9	− 9	− 1	+ 5	− 1
15	+31	·078	·29	−19	−13	− 8	− 5	−10
16	+20	·050	·16	−31	−36	−28	−36	−44
17	+ 8	·020	·03	−36	−25	−19	−32	−37
18	− 4	·010	·01	−16	+ 1	+ 3	+11	+13
19	−18	·045	·13	− 5	+ 2	+ 2	+12	+20
20	−30	·075	·27	+ 8	+21	+18	+19	+20
21	−44	·110	·39	+27	+20	+16	+15	+15
22	−57	·142	·43	+20	+ 8	+14	+ 6	− 1
23	−69	·172	·44	+17	+17	+22	+26	+19

TABLE VIII.

Tacubaya $-15°$.

Harmonic Analysis of the Residuals of Table VII.

log N_0.	sin θ.	cos θ.	sin 2θ.	cos 2θ.	sin 3θ.	cos 3θ.	sin 4θ.	cos 4θ.
0·62	+·075	+·119	−·053	+·106	−·061	−·023	+·018	−·027
1·22	+·038	+·137	−·044	+·054	−·041	−·019	+·058	+·014
1·76	·000	+·104	−·054	+·052	−·044	+·019	+·071	+·008
2·16	+·012	+·054	−·108	+·031	−·053	+·036	+·100	−·009
2·27	+·050	+·053	−·128	+·024	−·029	+·014	+·118	−·031

23. Confining attention for the moment to the first harmonic, the excess of the mean of the last two lines over that of the first two is

$$-·026 \sin\theta - ·074 \cos\theta = -·078 \cos(\theta - 19°).$$

The value of θ_0 for the "spiral" is $247^\circ + 55 = 302^\circ$, from which 19° differs by $+77^\circ$. It is perhaps worth remarking that a comparison of the last two lines with the *middle* line gives

$$+ \cdot 031 \sin\theta - \cdot 050 \cos\theta = -\cdot 059 \cos(\theta - 329^\circ),$$

which is in closer accord with the "spiral" value, but an adequate discussion of the residuals must wait until all the material has been cleared from the galactic phenomena.

24. A general revision of other zones, together with considerable new material, is proceeding, and the revised edition of the "spiral" will be given in a later communication.

On Baxendell's Diagram for the Variable S Vulpeculæ.

By J. G. Hagen, S.J., Assoc.R.A.S.

Observers of variable stars will be grateful to Professor Turner for publishing the "very interesting chart" reproduced in the ninth instalment of Baxendell's observations, plate 7 (*M.N.*, lxxvii. p. 555). The chart is indeed interesting, not so much for the variable *S* as for two groups of faint stars which clearly show that the significance of the diagram goes a good deal beyond its immediate purpose of observing *S Vulpeculæ*. These groups, which have nothing to do with the variable, might remind one of Pogson's charts, which likewise contain stars down to the thirteenth magnitude, even for bright variables. Yet the diagram is not one of Pogson's 34 printed charts, nor could it be, since Pogson left England for Madras in 1861, when Baxendell first established the periodicity of this variable.

1. The one group of small stars is close to the variable *S*. Its purpose is not the identification of that star, which remains within the range of 8·4 to 9·8 mag., and is, moreover, easily picked out by its orange tint (3·0 on Chandler's scale). Baxendell, in a note to be cited presently, calls it "one of the highly-coloured stars, both Mr. Hind and myself having always noted it as being *very red*."

The other and much larger group clusters around the star 3, and is evidently meant for some other object *np* the variable. The rectangle in which it is enclosed and the smallest star-numbers 1, 2, 3, 4, 5, falling within or near the rectangle, all point to this.

Fortunately, it is Baxendell himself who gives us the explanation. In a note which he read before the Manchester Literary and Philosophical Society in 1864 Oct. 18, he presents a small diagram made with the 13-inch reflector and exhibiting four companions *a*, *b*, *c*, *d* to No. 1773 Greenwich 12-year Catalogue (*S Vulpeculæ*). Three of them, *b*, *c*, *d*, he had found to vary in magnitude, and he concludes by saying: "we have here *four* variable stars within an area of little over one square minute."

Still more important is his additional remark: "that the place

of Anthelme's new star of 1670 precedes this singular and interesting group with a difference of right ascension of only about 48 seconds."

The position of Anthelme's Nova, or Flamsteed's 11 *Vulpeculæ*, was computed twice independently, by Hind in *Monthly Notices*, xxi. (1861) 231, and by Schönfeld in his Catalogues of Variable Stars (Mannheim, 1866, 1875), I. No. 93 and II. No. 115. The comparison of the two results will give an idea of their accuracy :

Hind : $19^h\ 41^m\ 39^s + 26°\ 57'{\cdot}3$ (1855).
Schönfeld : „ „ 37 „ 57'7 („).

The position of the Nova relative to *S Vulpeculæ* is therefore $-49^s + 1'{\cdot}8$ (1855) within $0'{\cdot}2$. Baxendell's star 3, as measured at Oxford, is so close to this place that it may possibly be the Nova, although Schönfeld remarks that the 11th mag. star has a faint companion. Hind believed he had established the variability of Baxendell 3, having seen it in mags. 10–11 and 12, and he was thereby led "to conclude that it is really Anthelme's *Nova Vulpeculæ*."

Should the variability of all these stars be confirmed, we should have five variables on the one diagram. But this is not all. Schönfeld suspected his own comparison star *g*, which is Baxendell's 9, to be slightly variable relative to a star *sp* it (A.S.V. No. 35) and just outside the diagram (see Schönfeld's I. Cat. No. 94, and *Astr. Nachr.* 1628–9). Now the very same star 9 declares itself variable with regard to star 10 in table iv., p. 559, of Professor Turner's and Miss Blagg's article. Thus the diagram may contain no less than six variable stars, one of which is perhaps the Nova of 1670. What Schönfeld wrote in 1866 (I. Cat. No. 94) of this region deserving careful and continued attention, remains true until the variability of the stars in question be eventually disproved.

2. The corresponding chart of the A.S.V. contains neither of the two groups of faint stars, for the reason that suspected variables and Novæ are on principle excluded from the Atlas. The chart is even relegated to series iv., which goes only as far as the 10th mag., a lower limit never reached by *S Vulpeculæ*. Still the chart may be of service in locating Baxendell's diagram in the heavens, as it covers nearly ten times as large an area.

The identification of the stars presents no difficulty, owing to the great accuracy of the diagram. The two charts are oriented alike, giving the inverted image of the sky. There are sixteen stars in common, as shown by the following table :—

Bax.	A.S.V.	Bax.	A.S.V.	Bax.	A.S.V.	Bax.	A.S.V.
1	var.	8	10	20	62	27	56
2	52	9	29	21	41	28	53
6	3	10	45	23 }	59 dpl.	30	48
7	49	12	61	25 }		37	60

3. It is not without interest that Baxendell, in a second note, read 1864 Dec. 8 before the same Society, reduced his own observations up to date. His epochs of maxima and minima are converted into Julian days to facilitate comparison with the epochs as given in the *Monthly Notices*. For the same reason the numbering (E) of the epochs in Chandler's Catalogue is retained throughout.

Maxima				Minima			
E	Bax.	Oxford	B.-O.	E	Bax.	Oxford	B.-O.
−12	2401434	1434	0	−12	2401469	1470	−1
−7	1761	1771	−10	−7	1804	1807	−3
−6	1829	1839	−10	−6	1865	1875	−10
−3	2038	2041	−3	−4	2005	2010	−5
2	2105	...	...	−3	2074	2077	−3
−1	2184	2176	+8	−2	2152	2145	+7

Baxendell's epochs are, on an average, 2·7 days earlier than the Oxford dates, which means simply a difference in the two zero points. Baxendell probably applied Pogson's bisecting curve, the use of which he had recommended the year before in a joint paper with George Knott ("Method of observing Variable Stars," London, 1863).

By a least square solution Baxendell obtained the elements:

Max. = 1864 Feb. 1·37 (2401903·37) + $67^{d}\cdot97$ (E + 5).
Min. = „ Jan. 1·59 (2401872·59) + 67 ·88 (E + 6).

As a mean result he adopts the period of 67·92 days. The ascending and descending branches of the light curve he makes 30·8 and 37·1 days respectively, and the range of variation 8·8 to 9·8 magnitudes.

Vatican Observatory:
1907 *Aug.* 15.

The Number and Galactic Distribution of the Stars.

By S. Chapman, M.A., D.Sc.

(*Communicated by the Astronomer Royal.*)

1. The present note is written in the hope that it may further the removal of the uncertainty which has, in the past, attached to the question of the number and distribution of the stars. The attempt is more particularly my duty, because an error of my own has assisted to prolong the uncertainty.

2. The discussion of the counts of stars on the Franklin Adams plates, published in 1914 by myself and Mr. Melotte

jointly,* appeared to show that the galactic condensation of the stars increases much more slowly, as the limit of faintness is extended, than Professor Kapteyn's investigation,† of 1908, had suggested. In this respect the new discussion seemed to support the conclusions arrived at still earlier (1903) by Professor E. C. Pickering.‡ The two earlier memoirs, however, had both dealt with the stars ordered according to their visual magnitude, while the C.-M. (Chapman and Melotte) discussion was based on photographic magnitudes. There remained a possibility, therefore, as Professor Turner pointed out, of reconciling Professor Kapteyn's and the C.-M. results, should it be found that the faint stars in the galaxy are increasingly bluer than those near the galactic poles.

3. In consequence of subsequent investigations by Dr. van Rhijn§ and Professor Seares,‖ these uncertainties may now be regarded as disposed of. This is far from implying that the matter has received the full investigation it requires: the accounts of their work, by Dr. van Rhijn and Professor Seares, are expressly preliminary to the publication of later and more complete discussions; but the agreement between their present results, and also between those and the results of the C.-M. counts, when newly reduced in a proper manner, leaves no doubt remaining upon the broad issue regarding the galactic condensation of the faint stars.

4. The position may be briefly summarised as follows:—The "galactic ratio" of the star densities N_0^{20}, N_{40}^{90} respectively, in the belt of 20° on either side of the galactic plane, and in the spherical caps of 50° radius around the galactic poles, is constant and equal to 2·5 for the stars down to magnitude m, where m ranges from 5 to 10 inclusive. From the latter magnitude the ratio steadily increases, its values corresponding to $m = 16$ and $m = 17{\cdot}6$¶ being respectively 5·5 (van Rhijn) and 10·2 (Seares). Kapteyn's values for the same *nominal* limiting magnitudes were 1·8 (5^m) to 2·6 (10^m), 10·4 (16^m) and 17·5 ($17^{m}{\cdot}6$); but it is necessary to add that the two latter are extrapolations from his data, and also that they refer to magnitudes which on the Harvard scale are numerically greater than those indicated. The magnitude scale used in Kapteyn's paper has proved, as a result of later research at the Harvard and Mount Wilson observatories, to diverge considerably from the absolute scale since determined. When this is allowed for, van Rhijn finds that Kapteyn's values

* *Mem. R.A.S.*, lx. p. 145.

† Kapteyn, *Groningen Publications*, No. 18.

‡ E. C. Pickering, *Harvard Annals*, xlviii. p. 149.

§ van Rhijn, *Groningen Publications*, No. 27, 1917.

‖ Seares, *Proc. Nat. Acad. Sci.*, iii. p. 217, 1917; Professor Seares has also (*Ap. J.*, xlvi. p. 117, 1917) discussed Professor Turner's counts of astrographic stars, with results which are in accordance with the others discussed in the present note.

¶ The magnitude scale used by Dr. van Rhijn and Professor Seares is that of the final North Polar sequence, *Harvard Annals*, lxxi. p. 231.

of the galactic ratio, while still increasing somewhat too rapidly with m, are much more nearly in agreement with his own than are the figures just quoted. The revised values increase from 2·1 (5^m) to 6·3 (16^m), and at $17^m{\cdot}6$ to 11·0.

These results indicate a marked increase of galactic condensation for the faint stars (10^m to 16^m), whereas the C.-M. memoir suggests only a small increase, from 2·1 at 5^m to 3·1 at 16^m. Dr. van Rhijn points out, however, that the method of reduction used in that investigation was faulty. The serious effect of this error is evident from the fact, so satisfactory in itself, that upon the application of the correct method the Franklin Adams counts yield results in close accordance with Dr. van Rhijn's (*cf.* § 9). All the recent investigations, therefore, are in general agreement as to the increase in galactic condensation for faint stars, and the conclusion may now be fairly regarded as established.

5. The last three discussions are all based on photographic methods and upon photographic magnitudes. That by Dr. van Rhijn deals with 65 Northern Durchmusterung plates (limiting magnitude about 16) of the Selected Areas,* taken at Harvard Observatory. Professor Seares' preliminary note relates to counts on 88 plates (limiting magnitude about 17·6) of the Selected Areas taken at Mount Wilson.† The C.-M. memoir was based on 30 magnitude plates (limiting magnitude from 16 to 17) taken at Greenwich, and upon the counts on the Franklin Adams plates, which were standardised by the former. The magnitudes of the stars on the Harvard plates were individually determined, the results being obtained to two decimal places; the stars within $0^m{\cdot}6$ of the limiting magnitude on a plate were not included in the counts. The Mount Wilson results, at present, refer only to total densities down to the limiting magnitude of the plates. In the Franklin Adams counts the individual star magnitudes were determined with much less accuracy than in the Groningen measurements of the Harvard plates; the revised results, however, show them to be satisfactorily free from any systematic error depending on magnitude or galactic latitude. Probably it would have been better, however, if the attempt had not been made to press the data too far by counting the stars down to the limiting magnitude on the plates; for this reason, in the present revision, the counts of stars fainter than those of magnitude 16 are excluded.

The original discussion of the Franklin Adams counts from 30 plates was intended to be preliminary to a fuller investigation based on the whole number of plates. Owing to the war and other causes, this fuller discussion has been suspended since 1914. It is perhaps doubtful whether it should be renewed (despite the proof that the counts will yield satisfactory statistical material), in view of the superior data which will be available on the completion

* Kapteyn, *Plan of Selected Areas*, 1906.

† An account of the present status of the Mount Wilson work on the Selected Areas is given by Professor Seares in *Proc. Nat. Acad. Sci.*, iii. p. 188, 1917.

of the catalogue of individual star magnitudes, down to a fainter limit, from the Mount Wilson plates of Selected Areas. On the latter account, also, this revision of the results in the C.-M. memoir will be confined within narrow limits.

6. In counting the stars on the Franklin Adams plates, 25 regions on each plate, at intervals of 3°, were dealt with, and about 60 stars were counted on each region (the areas counted being chosen accordingly). This was done in order to be able to get an equally good determination from each region of the *rate of increase* of star density down to successive limits of magnitude. The stars were classified by direct comparison, within the one eyepiece of the viewing microscope, with a scale of comparison stars. The classification thus effected was standardised by the determination of the true photographic magnitudes of the stars in the central region of each Franklin Adams plate, for which an auxiliary photometry plate was available; thirty such were at hand for discussion at the time when the C.-M. memoir was prepared. The auxiliary photometry plates were taken at Greenwich with the 30-inch reflector, and were exposed under similar conditions on the central field of the Franklin Adams plate, and on the field of the Harvard North Polar sequence. The magnitudes for the other 24 regions on each Franklin Adams plate were deduced from the calibration of the scale of comparison stars for the central region, by allowing for the falling off in apparent magnitude, with increasing distance from the plate centre. These position corrections were statistically inferred, and were found to be satisfactorily uniform for different groups of plates taken under similar conditions. The effective range of the counts was from the 12th to the 16th magnitude, though the attempt was made to deduce results down to the 17th magnitude, from some of the plates.

7. In reducing the counts, the results were modified, where necessary, so as to equalise, approximately, the number of stars down to magnitude 16 from each region, by adjusting the corresponding area. The sums of the number of stars, and of the areas, were then formed, and the former were divided by the latter in order to obtain the star density. This method gives a fair determination, over the magnitude range 12 to 16, of the mean rate of increase of star density with increasingly faint limit of magnitude, allowing equal weight to each region. The result obtained indicates that this mean rate does not differ very greatly from one galactic zone to another (*cf.* also § 15). On the other hand, since the sample regions are uniformly distributed, the method will not yield a correct value of the mean star density in any zone in which the star density is not constant. This fact is pointed out by Dr. van Rhijn, and when once noticed it seems surprising that the error was made, or, if made, not sooner discovered. I should add that the responsibility for the adoption of this faulty method of reduction was mine, and that Mr. Melotte, my collaborator in the work, was not concerned in it.

8. The mean star density in any zone is properly determined as the arithmetic mean of the star densities in sample areas uniformly distributed throughout the zone. The method actually adopted in the reductions afforded, on the other hand, approximately a harmonic mean of the densities; since the harmonic mean is always less than the arithmetic mean (in the case of positive quantities which are not all equal), the star densities thus obtained were in every case too low. The defect in value is the greater the more unequal the individual densities of which the mean is taken; now the range in density is notably greater in the galaxy than near the galactic poles, so that, while the results for the higher galactic zones are little below their proper values, those for the galaxy itself are appreciably reduced. The galactic condensation for the stars of magnitudes 12 to 16 was thus minimised. It should be mentioned, in this connection, that the results for the stars brighter than 12^m were not affected by the above error, since they were obtained differently, and from other sources of data.

9. The detailed counts, and the corresponding areas, on the 30 Franklin Adams plates were not published in the C.-M. memoir; but from the original manuscripts it is a very simple matter to revise them so as to place the results on a proper equal-area basis for each region. The logarithms of the star densities so obtained are given in Table A, for each whole magnitude from 12 to 16, on the scale of *Harvard Circular*, No. 170. These log densities correspond to those in table iii., p. 152, of the original memoir (but note that, as regards zone VI., 4·67 should there be replaced by 5·89, and 0·10 or 0·11 subtracted from the first nine values of log B_m); both are directly deduced, without any smoothing, from the original counts. In Table A, alongside the revised C.-M. results are placed the similarly unsmoothed results from Dr. van Rhijn's memoir, which, however, extend only to magnitude 15·5; the magnitude scales for the two sets of results are nearly but not quite the same (§ 10). The zones I. to VIII. range from the galaxy to the galactic poles, by 10° belts on either side of the galaxy, except that zone VIII. extends from latitudes ± 70° to the galactic poles.

TABLE A.

Zone.	No. of Gauges.	Log Density. m. = 12.		m. = 13.		m. = 14.		m. = 15.		15·5.	16.
		C.-M.	v. R.	C.-M.	v. R.	C.-M.	v. R.	C.-M.	v. R.	v. R.	C.-M.
I.	187	2·00	1·83	2·30	2·22	2·69	2·54	3·01	2·92	3·14	3·28
II.	102	1·85	1·76	2·14	2·16	2·51	2·48	2·84	2·83	2·99	3·18
III.	109	1·75	1·63	2·07	1·96	2·37	2·25	2·67	2·61	2·77	3·00
IV.	133	1·69	1·51	1·99	1·79	2·23	2·11	2·55	2·40	...	2·87
V.	111	1·49	1·38	1·79	1·67	2·05	1·97	2·33	2·28	...	2·63
VI.	32	1·43	1·27	1·70	1·58	1·97	1·85	2·17	2·17	2·31	2·56
VII.	35	1·16	1·25	1·65	1·54	2·03	1·82	2·27	2·10	2·26	...
VIII.	41	1·32	1·17	1·62	1·52	1·89	1·78	2·14	2·04	2·16	...

The differences between the two sets of results in Table A are given in Table B (the C.-M. values of log density for magnitude 15·5 were obtained by simple interpolation between the values for m. = 15 and m. = 16). The table also exhibits the mean differences for each magnitude (all zones) and for each zone (all magnitudes). The latter mean difference shows little or no systematic progression, but only a constant difference affected by accidental error. If the

TABLE B.

Zone.	Differences, C.-M.—v. R. (Table A).					
	m. = 12.	m. = 13.	m. = 14.	m. = 15.	m. = 16.	Mean.
I.	+0·17	+0·08	+0·15	+0·09	+0·00	+0·10
II.	+0·09	−0·02	+0·03	+0·01	+0·02	+0·03
III.	+0·12	+0·21	+0·12	+0·06	+0·07	+0·12
IV.	+0·18	+0·20	+0·12	+0·15	...	+0·16
V.	+0·11	+0·12	+0·08	+0·05	...	+0·09
VI.	+0·16	+0·12	+0·12	+0·00	+0·05	+0·09
VII.	−0·09	+0·11	+0·21	+0·17	...	+0·12
VIII.	+0·05	+0·10	+0·11	+0·10	...	+0·09
Mean	+0·10	+0·11	+0·12	+0·08	+0·04	...

latter were eliminated by smoothing, as would be done in deducing final values in a complete reduction, the galactic condensations derived from the Franklin Adams plates and from Dr. van Rhijn's study would be practically identical. The mean differences according to magnitude (without reference to galactic latitude) perhaps show a slight progression, though if the last value 0·04, obtained from only half the zones, were omitted, this would hardly be suggested. On the whole, the comparison of the results in Table A indicates little more than a mean difference of 0·095 between the log densities (C.-M.–v. R.).

10. This systematic difference may be partly accounted for by the difference between the magnitude scales underlying the two discussions. Miss Leavitt* has compared the provisional and the final scales of the North Polar standard magnitude sequence, and it appears that the difference is constant, and equal to $+0^{m}\cdot14$, between magnitudes 11 and 21. This small difference of zero-point would naturally produce an almost constant difference between the log densities over the small magnitude range (12 to 16) involved in Table A. The values of the increase of log density for unit increase in the limiting magnitude are (from 11·5 to 12·5, 12·5 to 13·5, and so on) 0·38, 0·34, 0·33, 0·31 respectively; hence the differences in the log densities down to

* *Harvard Annals*, lxxi. p. 224; the provisional scale, used in the C.-M. memoir, was published in *Harvard Circular*, No. 170; the final scale, used by Dr. van Rhijn and Professor Seares, is described in *Harvard Annals*, lxxi. p. 231.

magnitudes 12, 13, 14, 15, due to a difference of 0·14 in the zero-points of the magnitude scales, should be 0·053, 0·048, 0·046, 0·043 respectively. The sign of the differences is the same as that of those in Table B. The mean of the numbers just given is 0·047, so that half the observed difference of 0·095 is thus explained. The remaining difference, of course, is therefore equivalent to a further zero-point discordance of $0^{m}\cdot14$ between the two magnitude scales. Whether or not this is the cause of the discrepancy, it is not easy to say; the difference may perhaps be explicable rather as a consequence of the fact that the two sets of data do not apply to the same sample areas (*cf.* § 15 for instances of the range of density within individual galactic zones).

11. It is of interest also to compare the revised C.-M. results in Table A with the numbers originally given (*cf.* § 9). The following table enables this to be done. It appears that the revision scarcely affects the results for the higher galactic zones, while in the galaxy the numbers are increased by amounts which are perhaps

TABLE C.

Zone.	Differences, C.-M. (rev.)—C.-M. (orig.).					
	m. = 12.	m. = 13.	m. = 14.	m. = 15.	m. = 16.	Mean.
I.	0·13	0·18	0·20	0·20	0·17	0·18
II.	0·13	0·12	0·14	0·14	0·15	0·14
III.	0·05	0·09	0·04	0·04	0·08	0·06
IV.	0·06	0·07	0·06	0·09	0·16	0·09
V.	0·01	0·07	0·05	0·06	0·06	0·05
VI.	0·01	0·00	0·02	0·01	0·02	0·01
VII.	0·03	0·05	0·03	0·03	...	0·03
VIII.	0·03	0·06	0·01	0·00	...	0·02
Mean	0·06	0·08	0·09	0·07	0·11	...

the greater at the fainter limit, and in all cases are considerable. These circumstances are natural, in view of the great range in star density in the Milky Way, and the small range near the galactic poles. The latter feature is well illustrated by the numbers in columns 3, 4 of table xvii. (C.-M.).

12. In view of these consequences of the change in the method of grouping the Franklin Adams counts, I thought it desirable to examine whether the reduction of the magnitude scale on each plate, from the centre to the 24 outer regions, would be at all affected by the same cause. The 74 plates used in deducing the plate-position corrections were disposed without any particular relation to the galaxy, so that, the reduction being differential, no influence of the suggested kind would be expected, but it seemed well to make sure as to the point. Accordingly the counts on the Northern plates taken at Mervel Hill (76 in number) were re-discussed; the $25 \times 76 (= 1900)$ regions counted were collected into

four groups, *i.e.* 76 central regions, 4×76 regions 3° from the centre, 4×76 regions $3\sqrt{2}$° from the centre, and the 16×76 regions in the outer square of 12° side. These four groups are the same as those referred to in table i., C.-M., where, however, the order of the column headings should be reversed. The total numbers of stars in each region, per 100 square minutes of arc, down to the limiting magnitudes M_9, W_9, X_9, Y_9, and Z+ (*cf.* § 5, C.-M.), were formed for each group of plate regions. In the C.-M. memoir the numbers actually counted, for regions of different areas, were added up. After dividing the new sums by the number of regions in the corresponding group, the logarithms of the resulting numbers give comparative measures of the log densities down to the given limits of magnitude, at the four different distances from the plate centre. The differences between the values at the centre and those at the other distances give the position corrections to be made (in whatever way seems most convenient) to the star densities deduced for the outer groups. The following table shows the results obtained by the new reduction (N.R.) and by the old (C.-M.).

TABLE D.

Position-corrections (to centre) for the Log Densities from different Regions on the Franklin Adams' Plates.

Magnitude:—		M9.	W9.	X9.	Y9.	Z+
3°	N. R.	0·04	0·06	0·07	0·05	0·06
	C.-M.	0·06	0·08	0·08	0·05	0·06
$3\sqrt{2}$°	N. R.	0·16	0·15	0·14	0·11	0·11
	C.-M.	0·15	0·14	0·13	0·11	0·10
Outer square	N. R.	0·30	0·30	0·29	0·20	0·20
	C.-M.	0·27	0·26	0·28	0·20	0·19

The differences are clearly very small, and not systematic in sign for the different regions. There seems no ground, consequently, for thinking that the unexplained difference of 0·047 between the Groningen-Harvard and the Franklin Adams results is due to the plate-position corrections used in reducing the latter.

13. In the appendix to the C.-M. memoir a table (xvi.) was given showing the total numbers of star images estimated to be enumerable in the central square of 15° side on each of the Franklin Adams plates. This estimate was formed from the 25 sample counts on each plate, giving equal weight to the total star *density* from each sample, in the proper manner. In table xvii. these star totals were grouped, according to galactic latitude, in four zones. The mean densities in each zone, assuming an approximately uniform limiting magnitude for the various plates (and this assumption seems legitimate, at any rate for the plates taken at Johannesburg), appeared to confirm the conclusion as to the galactic condensation, which was stated in the memoir proper. Since reading Dr. van Rhijn's memoir I have examined this circumstance with consider-

able care, and as a result I now feel no doubt that it is due to incomplete enumeration of the stars in the denser regions of the sky. The attempt was made to count all the stars, down to the faintest visible, in each region, but it was clearly not successfully accomplished in the dense regions; the limiting magnitude corresponding to the galactic counts in table xvii. is therefore not equal to that for the counts in higher galactic latitudes. This fact, of course, does not at all affect the results in the main part of the C.-M. memoir, which do not rest upon any assumption as to the limiting magnitudes of the plates.

14. The present opportunity may be taken of bringing forward further evidence obtained at Greenwich with regard to the number and galactic distribution of the stars. This is afforded by counts of stars of determined photographic magnitude situated in the Selected Areas and in the central areas of the Franklin Adams plates. The photographs on which the determinations rest were taken with the 26-inch refractor, and the results are independent of those from the plates taken with the 30-inch reflector, which provided the magnitude basis for the C.-M. memoir. The star counts on the new plates may be relied upon as complete to magnitude 13 at least. The magnitude scale is the same as that used for the Franklin Adams counts, viz. that of *H.C.* 170. The counts relate to a square of 40′ side in every case; they were made down to limiting magnitudes 10·99, 11·49, 11·99, 12·49, and 12·99, or, briefly, to 11·0, 11·5, and so on. The results from 16 Franklin Adams areas in zone +15° declination, and 38 Kapteyn areas, were combined into three groups of galactic latitudes (±) 0° to 20°, 20° to 40°, and 40° to 90°. The numbers of plates in each of these groups were respectively 15, 24, and 15. The mean values of log density so obtained, and also similar results extending (save for the third zone) to the 16th magnitude—obtained from the revised C.-M. results in Table A,—are given below, side by side with Dr. van Rhijn's final smoothed results from his table ii.

TABLE E.

Limit of Mag.	Log N_0^{20}.			Log N_{20}^{40}.			Log N_{40}^{90}.		
	v. R.	Gr. 26 in.	C.-M.	v. R.	Gr. 26 in.	C.-M.	v. R.	Gr. 26 in.	C.-M.
11·0	1·38	1·50	...	1·16	1·20	...	0·92	0·95	...
11·5	1·59	1·66	...	1·36	1·41	...	1·11	1·15	...
12·0	1·80	1·86	1·93	1·58	1·54	1·72	1·27	1·29	1·37
12·5	1·99	2·04	...	1·73	1·70	...	1·44	1·45	...
13·0	2·18	2·26	2·23	1·90	1·90	2·03	1·59	1·62	1·69
14·0	2·53	...	2·61	2·21	...	2·31	1·88	...	1·99
15·0	2·86	...	2·94	2·52	...	2·62	2·16	...	2·23
16·0	3·17	...	3·23	2·81	...	2·95	2·43	...	...

The differences between the second or third columns and the first, for each group, are given in the next table.

TABLE F.

Limit of Mag.	Log N_0^{20}.		Log N_{20}^{40}.		Log N_{40}^{90}.		Mean.	
	Gr.–v. R.	C.-M.–v. R.	Gr.–v. R.	C.-M.–v. R.	Gr.–v. R.	C.M.–v. R.	Gr.–v. R.	C.M.–v. R.
11·0	+0·12	...	+0·04	...	+0·03	...	+0·06	...
11·5	+0·07	...	+0·05	...	+0·04	...	+0·04	...
12·0	+0·06	+0·13	−0·04	+0·14	+0·02	+0·10	+0·01	+0·12
12·5	+0·05	...	−0·03	...	+0·01	...	+0·01	...
13·0	+0·08	+0·05	+0·00	+0·13	+0·03	+0·10	+0·04	+0·09
14·0	...	+0·08	...	+0·10	...	+0·11	...	+0·10
15·0	...	+0·08	...	+0·10	...	+0·07	...	+0·08
16·0	...	+0·06	...	+0·14	...	...	...	+0·10
Mean	+0·08	+0·08	0·00	+0·12	+0·03	+0·09	+0·03	+0·10

It appears from this table that the *range* of the differences in columns 1 (Gr.–v. R.) and 2 (C.-M.–v. R.) of the various groups is about the same, but that the mean difference is less in the former than in the latter case. In the former case, indeed, the small mean difference 0·035 is fully explicable by the difference of zero-point between the magnitude scales of *H.C.* 170 (Gr. 26-inch) and *H.A.* 71 (v. R.). Apart from these small systematic differences, the three sets of results in Table E are in fair agreement.

15. A question of interest which arises from the revision of the C.-M. results relates to a possible difference in the rate of increase of star density with diminishing limiting brightness, in regions of different star density within the same galactic zone. The excess weight given to the results from the least dense regions of the galaxy, by the original method of reduction of the counts, led to too small a rate of increase of star density with magnitude. This suggests that the rate of increase in those regions is less than for the denser regions. In order to throw further light upon this point, I have subdivided the results from zones 0° to 20°, 0° to −20°, ±20° to ±30°, and ±30° to ±40° galactic latitude, so as to separate the counts for the denser and rarer regions. In the case of the first two zones three groups were taken, containing respectively the regions where there were 80 or more stars, down to magnitude 16, per 100 square minutes of arc, and those where there were similarly from 50 to 79, and less than 50. In the case of the third and fourth zones, only two groups were considered, the corresponding limits of star numbers being 35 and 21 respectively. The results for the log density (log N_m), for the magnitudes 12 to 16, are given below (Table G), together with the number of areas from which each result is obtained. The figures indicate that the increase in log N_m from $m = 12$ to $m = 16$ is distinctly greater for the richer regions in the galaxy than for the less dense regions; the rate of increase in the latter agrees with that found at a considerably greater galactic latitude. In zones

III. and IV. little or no such differences appear to exist. After these figures had been obtained I separated from the first two zones the results for the central galactic belt from $-10°$ to $+10°$ galactic latitude; it may be noted that 33 out of the 35 densest regions in those two zones come within the central belt. The results for the same three groups of regions within this belt are placed first in Table G; they show in a still more marked manner the feature already remarked in the other zones. If this evidence for the phenomenon is confirmed, a fact of some significance will be added to our knowledge of galactic structure.

TABLE G.

Zone.	Density and (Number of Regions).	Log N_m. m.=12.	m.=13.	m.=14.	m.=15.	m.=16.	Increase from 12 to 16.
I.	Great (33)	2·25	2·63	3·07	3·37	3·60	1·35
	Moderate (57)	2·05	2·33	2·75	3·08	3·35	1·30
	Small (96)	1·83	2·14	2·34	2·68	3·01	1·18
0° to 20°	Great (15)	2·27	2·65	3·10	3·40	3·58	1·31
	Moderate (46)	2·04	2·38	2·80	3·10	3·35	1·31
	Small (96)	1·76	2·03	2·36	2·66	2·97	1·21
0° to −20°	Great (20)	2·21	2·59	3·04	3·34	3·59	1·38
	Moderate (51)	1·99	2·24	2·62	3·00	3·35	1·36
	Small (60)	1·85	2·03	2·30	2·70	3·05	1·20
III.	Greater (19)	1·98	2·22	2·54	2·85	3·22	1·24
	Smaller (75)	1·56	1·88	2·22	2·52	2·85	1·29
IV.	Greater (28)	1·79	2·07	2·30	2·61	2·96	1·17
	Smaller (75)	1·54	1·83	2·08	2·36	2·68	1·14

16. A few remarks may be added, in conclusion, upon the subject of the mean star density over the whole sky, for different magnitude limits. The original method of reduction of the Franklin Adams counts made the resulting estimates of mean density n_m slightly too small. The difference of zero-point in the magnitude scales tends in the opposite direction, when the original C.-M. results are compared with those of Dr. van Rhijn. This is indicated in the following table (the C.-M. values are taken from column 2 of table xiv., after deducting 4·62, the logarithm of 41,250, the number of degrees in the whole sky; the v. R. values are taken from table ii., *loc. cit.*):—

TABLE H.

	Log n_m. m.=7.	m.=10.	m.=13.	m.=16.
C.-M. (original)	$\bar{1}$·37	0·81	1·95	2·85
v. R. (table ii.)	$\bar{1}$·34	0·75	1·95	2·90

The new reduction of the Franklin Adams counts leads, as appears from Table B, to values of log n_m which are approximately 0·05 larger than those of Dr. van Rhijn's paper (over the magnitude range 12 to 16), when allowance is made for the difference of zero-point of the magnitude scales used in Table A.

In the C.-M. memoir an attempt was made to extrapolate beyond the observed results, so as, if possible, to obtain some estimate as to the probable total number of stars. Four estimates were given, that numbered II. being referred to as the most probable; it assigned 1100 millions as the total number of the stars, of which half were brighter, and half fainter, than magnitude 23·3. The other three estimates gave the values $760 . 10^6$ and 22·5, $1800 . 10^6$ and 24·3, $3000 . 10^6$ and 25·4 for the same two quantities. Dr. van Rhijn makes a similar calculation, and obtains the results $3360 . 10^6$ and 25·5. As Table H shows (*cf.* also § 15 of this paper), the original reduction of the Franklin Adams counts led to a rate of increase of star density with magnitude which was slightly smaller than that which the data really indicate, or than that obtained by Dr. van Rhijn. This explains why the latter's estimate of the total number of stars, and the median magnitude, is rather larger than even the greatest estimate given in the C.-M. memoir. These estimates of the total number of the stars are mentioned, not as suggesting that any special weight can be attached to them, since they are admittedly greatly extrapolated, but as indicating the degree of agreement between more moderate extrapolations from the two sources of data under discussion. From this point of view the agreement may be regarded as not unsatisfactory, when the large uncertainties obtaining previous to the publication of the Franklin Adams counts are considered.

The Spectrum of Planets Jupiter and Saturn. By Professor L. Becker, Ph.D. (Plate 1.)

The spectrograph with which the planets were photographed has been described in a former paper.* Its film-holder is curved to the focal surface of the camera, and hereby the spectrum can be photographed in perfect focus over its whole length, about 11 cm., from wave-length 370 to 706. The film-holder is mounted on a micrometer slide, and its position can be adjusted on a spectral line. The effect of flexure is thus avoided.

In this paper I have utilised only two photographs of each planet. Those of Jupiter contain also the spectrum of the moon when at about the same altitude as the planet. The iron-sodium arc furnishes the comparison spectrum. The spectra of the planets are shown on the films from the B to the K lines.

There are several absorption bands (see No. 1 on Plate) due to

* *M.N.*, vol. lxxii. p. 709.

the dyes employed in sensitising the film for the less refrangible rays, and the maximum of one of these bands almost coincides with the dark band peculiar to the planets, and apparently displaces and broadens the planetary band (compare Nos. 2 and 6 on Plate). In order to eliminate this disturbing effect of the dyes, and also that due to the grouping of the fainter lines of solar or terrestrial origin, I made, in the case of a Jupiter film, an enlarged positive on glass from the film negative, and next I made a contact negative of the lunar spectrum from this glass positive. When these two corresponding positive and negative glass plates are superposed face to face, so that the positive lunar spectrum covers the negative spectra of moon and Jupiter, the two superposed lunar spectra present the appearance of a continuous spectrum of uniform intensity on which the dye-bands and also the fainter solar and telluric lines have disappeared (see No. 5 on Plate). The spectrum of Jupiter with the superposed positive lunar spectrum differs from the latter only by the bands peculiar to the planet's atmosphere (see No. 6 on Plate). I photographed the two superposed plates in the camera with transmitted light, and measured the position of the planetary band on this photograph. This copying process was repeated several times, each of the two films of Jupiter furnishing six photographs for measurements.

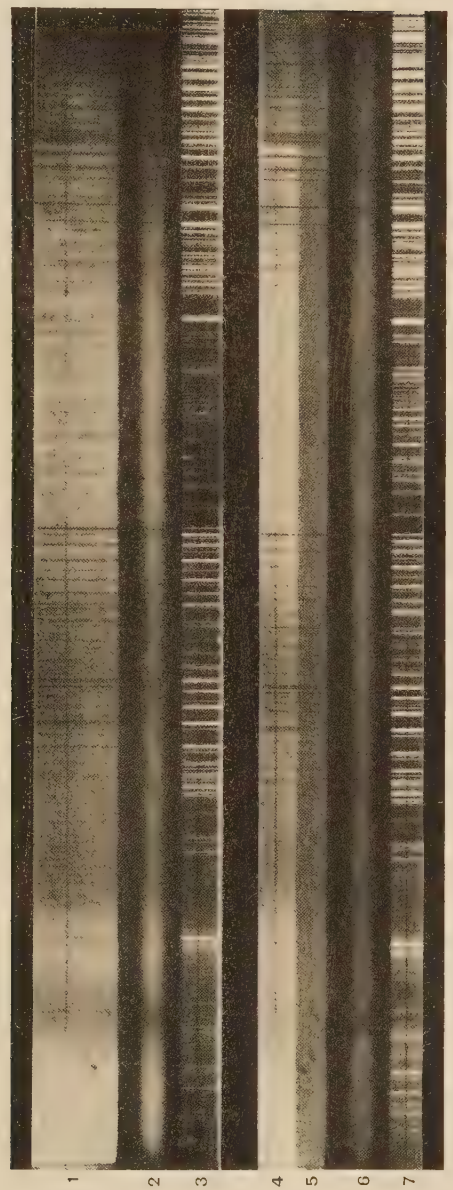
The films with the spectrum of Saturn were treated in the same way, but with this difference, that the negatives of the lunar spectrum as copied from the Jupiter plates were employed.

There is only one band discernible in the spectra of the planets. The Jupiter film taken 1909 April 30, with an exposure of 3 hours, gives its position at wave-lengths 6175 to 6207, and the film exposed during $2\frac{1}{4}$ hours on 1909 May 6 gives the figures 6174 to 6207. The Saturn films give the figures 6173 to 6216 and 6173 to 6218. The former film was exposed during 12 hours on three days, 1911 November 20, 21, 22, and the latter during 7 hours also on three days, 1912 February 20, March 5 and 6. I adopt the position 6174 to 6214.

Water vapour has been supposed to be present in the atmospheres of these planets, but this supposition is not borne out by our films. For comparing the intensities of the water-vapour lines in the rain-band, near the D lines, with those appearing in the lunar spectrum, I find that they are faint in every case and do not differ in appearance in the spectra of moon and planets. Since there are no lines photographed beyond wave-length 706, the *a* band caused by water vapour does not appear on the plates.

On the accompanying Plate the spectra Nos. 1, 4, and 5 belong to the moon, Nos. 2 and 6 belong to Jupiter, and Nos. 3 and 7 to the iron-sodium arc. Nos. 1, 2, and 3 are reproductions of the photograph taken 1909 April 30, and Nos. 4 to 7 represent the same photograph, but with the reversed lunar spectrum superposed at Nos. 5 and 6.

Glasgow University Observatory:
1917 *November* 1.



Spectra of the Planet Jupiter, the Moon, and the Iron-Sodium Arc.

Geographical Conditions for the Observation of the Total Solar Eclipse, 1919 *May* 28–29. By Arthur R. Hinks, M.A., F.R.S., Secretary Roy. Geog. Soc.

At the request of the Astronomer Royal I have tried to collect information which may serve as a guide in the selection of stations for the observation of this important eclipse. As I mentioned in a verbal communication made to the Society after the paper by the Astronomer Royal in March last, the conditions along the greater part of the track are unfavourable, since it lies largely in the two largest tropical river-basins of the world, the Amazon and the Congo. Possible stations would seem to be in Ceará, Brazil; the Island of Principe; in the neighbourhood of Libreville, French Congo; and on the western escarpment of Lake Tanganyika.

St. Paul's Rocks in the Western Atlantic are near the central line, but are quite unsuitable for the establishment of a station. They are described in the narrative Report of the *Challenger* Expedition as volcanic, with no fresh water except in the rainy season, and no anchorage. The central line cuts the coast of Liberia a few miles inside Cape Palmas (the diagram in the *Nautical Almanac* is incorrect here), but the rainfall at Cape Palmas is about 100 inches, the dry season is short, from December to February, and during the rains much of the coast is under water. Putting aside these two localities, we may briefly consider the other.

1. The interior plateau of the State of Ceará is described as quite healthy, and with less rain than on the coast or in the Amazon basin. The town of Sobral is about ten miles south of the central line, and Santa Anna is almost upon it. These have connection by railway with the port of Camocim. The rainy season is described as lasting from February to June, but the rains are intermittent, and sometimes fail altogether. The experience of the observers of the eclipse of 1893 was not unfavourable. I am indebted to Mr. J. C. Willis, formerly Director of the Jardin Botanico, Rio de Janeiro, for the information that the province of Ceará has a dry and sunshiny climate, and that no better place to see the eclipse can be desired. He suggests that inquiries should be addressed to Dr. Alberto Löfgren, of the Jardin Botanico, Rio de Janeiro, a well-known botanist who has an extensive knowledge of the State of Ceará. In accordance with this suggestion, a letter has been written to Dr. Löfgren, but up to the present no reply has been received.

2. The southern end of the Island of Principe lies some twenty miles north of the central line, but in many respects it offers great facilities for the establishment of an expedition, having good landing jetties, light railways, and large plantation buildings, especially in the north of the island. I am indebted to Captain Ernesto de Vasconcellos, Secretary of the Lisbon Geographical Society, for the following information:—

The largest company of planters in the island is the Sociedade de Agricultura Colonial, who own a very large property stretching south-west from Santo Antonio so far as Punto do Grosso, the south-west point of the island, in latitude 1° 32½′ N. Captain de Vasconcellos undertakes that an expedition would receive all facilities from this Society. He sends a copy of meteorological observations taken at the "observatory of the plantation" at a height of 135 metres, but in an unspecified locality, giving the following particulars for May 1914 and 1916:—

May 1914:	Clear sky . .	28 days
	Cloudy . .	1 day
	Heavy rain . .	2 days
May 1916:	Clear sky . . .	19 days
	Showers . .	7 ,,
	Heavy rain . .	4 ,,
	Thunder . .	1 day.

The temperatures in the house in these months were exceedingly uniform, about 26° centigrade, with very slight variation. Captain de Vasconcellos, who knows the island personally, suggests that the south-east portion would form the best station for observation. He is sure that the Board of Directors of the Sociedade de Agricultura Colonial will send orders to the manager of the plantation for accurate observations of sun and cloud this year and next.

3. The line of totality passes eighteen miles north of Libreville, the capital of French Equatorial Africa, at the mouth of the Gaboon River. There are hills a little way inland, north-east of the town, which might provide good stations. The rainy season is coming to an end in May. I am not able at present to give local information as to conditions here.

4. The station which seems to offer by far the best chances of clear sky is at Baudouinville, or in its neighbourhood, on the high western shore of Lake Tanganyika. In reply to inquiries addressed to Monseigneur le Vicaire Apostolique du Haut Congo, the following information has been received:—

Baudouinville and Lusaka, two important stations of the White Fathers, lie on the slope towards the lake at a height of about 1200 metres above sea, and about 350 above the lake. The lake shore is everywhere very steep, cut up by ravines and torrents, and the tracks are unbridged. Packages weighing as much as 100 kilograms would be very difficult to transport. The Mission has never attempted to transport so heavy a package, and has no experience as to the difficulties. West of these stations the ground continues to rise to the watershed between the short streams running down the Tanganyika escarpment and the Congo basin to the west. This watershed is in general about 2000 metres above sea, with occasional higher summits, some of them with flat tops. These summits would probably give a clearer western

horizon, but it would need a very strong party to establish an eclipse camp at such a height and in such a country.

Unless Baudouinville and Lusaka are much more shut in to the west than appears from the very imperfect maps available, it would probably be inadvisable to go further west than these stations.

The chances of clear sky on May 29 are very great. The rains ordinarily finish between the 1st and the 10th of the month, or, as the Bishop puts it, "with the April moon," and they rarely last beyond the 15th, though there may be some passing rain after that date. But during the twenty-five years that he has been in the country he has never seen the rainy season prolonged beyond May 15 in the neighbourhood of Baudouinville and Lusaka. This would give about fifteen days at least for preparation before the date of the eclipse. Casual cloud is possible in the morning, but rare in the evening. There is always a little mist in the air, which covers the whole country as soon as the dry season begins, and which limits the view to about 50 or 60 kilometres, whereas in the wet season one could sometimes see 150; but in May this mist is much less thick than later on in the dry season. In any case, even if the estimate that even in the beginning of the dry season one can expect to see 50 kilometres is not too sanguine—and this I should fear,—there can be little reason to hesitate in choosing Baudouinville or Lusaka as stations with every probability of fine weather. The Bishop promises to send rainfall observations from Baudouinville, and to have observations made at Lusaka. He very kindly concludes his letter by undertaking that his missionaries will be ready to give to an eclipse expedition all the assistance in their power.

We have also letters from Sir Alfred Sharpe, K.C.M.G., formerly Governor of Nyasaland, which give useful information and advice. He writes: "I should say that on the high mountain behind (west of) Baudouinville, on Lake Tanganyika, a station should certainly be established. The end of May would be about the very best season of the year for observation there, as the rains finish by the end of April; and though the dry season will have set in, the grass fires will not have commenced, and the air should be clear, with skies of open blue, with flecks of small fleecy clouds high up—the dry-season sky. I know the lake, but have not landed at Baudouinville. There is very high country west of the settlement: healthy. Communications would be good: mail from Dar-es-Salaam to Ujiji and steamer thence to Baudouinville. Plenty of native porters should be available at Baudouinville." In a later letter Sir Alfred Sharpe says: "Should there be an expedition from England to Tanganyika to observe this eclipse, and should the R.A.S. be in want of someone who knows Africa (and Tanganyika) to organise the transport matters in the country and carry out arrangements with native carriers, etc., I should be glad to give my services. We may hope that the war will be over before the beginning of 1919, in which case transport as

far as Ujiji would be quite a simple matter. There will probably, however, be a lot to do at Baudouinville in transporting everything to the site selected, in the erection of temporary buildings and shelters, in arrangements for supply of native food, and so forth."

Such is the information which I have been able up to the present to collect for the use of the permanent Eclipse Committee. It is fortunate that the conditions at Baudouinville, the most accessible of the high and healthy stations in Africa, seem to promise so well.

[*Note added* 1917 *November* 28.—The country west of Lake Tanganyika is not topographically surveyed, but geodetic triangulation has been carried to the northern end of the lake from a side of the Uganda-Congo Survey triangulation, while the southern end of the lake has been connected with boundary survey triangulations, and a line of triangles has been carried up the west coast to the Lukuga. The position of the mission town at Baudouinville is Lat. S. 7° 3′ 28″·79, Long. E. 29° 43′ 55″·0. We have at the R.G.S. a list of positions of many other points, communicated by Captain Maury of the Belgian Colonial Service.]

Micrometrical Measures of 250 *Double Stars.* (List V.)

By W. S. Franks.

PART I.

Σ.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
4	0 4·7	+ 7 54	8·7, 8·8	273·0	5·19	1916·811	2	
6 *rej.*	,, 5·1	4 18	9·2, 10·0	192·6	20·36	·806	2	Fr. 8·8, 10·0; D>s
8	,, 6·4	− 3 37	7·2, 8·8	292·2	7·58	·806	2	p. yel, bl.
22	,, 12·3	+ 8 19	Fr. 6·8, 7·7	237·6	4·55	·874	3	38 Pisc.; ylsh, bls wh.
24	,, 13·4	25 37	7·2, 8·0	248·5	5·14	·805	2	wh, wh.
[OΣ 5]	,, 13·5	43 14	6·5, 10·2	241·2	6·17	·805	2	26 Androm.; wh, o
34	,, 26·4	77 34	8·7, 8·8	336·2	6·07	·811	2	
44	,, 33·0	40 26	8·3, 9·0	268·3	10·02	·814	2	A and D<s.
48	,, 36·4	70 48	7·0, 7·2	332·8	5·39	·811	2	v. wh, v. wh; Lew +20° 48′.
61	,, 44·5	27 10	Fr. 6·3, 6·5	117·2	4·55	·817	3	65 Pisc.; both yls wh.
87	1 0·2	14 55	8·0, 8·5	198·7	6·53	·814	2	A<s?
[OΣ 22]	,, 1·8	11 1	7·2, 10·2	199·5	8·43	·814	2	p. yel, o; A<s.
95	,, 5·4	− 5 30	8·5, 9·7	309·7	13·97	·814	2	Fr. 8·7, 9·7.
99	,, 8·3	+24 3	4·7, 10·1	226·8	7·86	·840	2	φ Pisc.; orge yel, bl
101	,, 9·2	− 8 7	7·5, 9·8	344·7	20·60	·840	2	A<; yel, blsh.

PART I.—*continued.*

Σ	R.A. 1900. h m	Decl. 1900. ° ′	Magnitudes.		Pos. Ang. °	Distance. ″	Epoch.	Nts.	Remarks.
129	1 24·9	+12 8	8·5,	9·0	282·5	8·72	1916·840	2	
132	,, 26·7	16 27	7·0,	10·0	347·6	41·48	·828	2	yel, o.; A>D<.
172	,, 44·4	26 35	9·0,	9·2	194·4	17·48	·828	2	
200	,, 56·0	23 37	8·5,	9·0	123·8	7·98	·840	2	
202	,, 56·9	2 17	Fr.	4·0,	5·0	310·5	2·53	·874	3	*a* Pisc.; gnsh wh, fawn.
215	2 2·9	40 19	8·2,	9·7	58·8	20·20	·852	2	D<s?
256	,, 17·1	48 53	8·2,	9·5	195·8	21·00	·852	2	AB.
,,	,, ,,	,, ,,	,,	9·5	44·0	36·74	·852	2	AC. Fr. C = 8·5; 1^{m}> B; var.? } Triple.
282	,, 32·8	65 12	8·3,	8·3	114·2	7·02	·852	2	
295	,, 36·1	− 1 7	6·0,	9·2	319·2	4·66	·852	2	84 Ceti; wh, prp; A>.
297	,, 38·1	+56 8	8·0,	8·3	277·7	16·39	·852	2	AB. D<s?
301	,, 40·6	53 30	7·3,	8·3	16·4	8·12	·852	2	yel, bl.
316	,, 45·8	36 53	8·5,	8·7	134·4	14·38	·858	2	
321	,, 48·7	58 28	8·5,	9·0	20·8	18·43	·858	2	
325	,, 49·6	34 4	8·2,	9·7	187·2	9·82	·866	2	A>D<; Lewis 7·5, 7·5.
330	,, 52·1	− 1 0	7·5,	9·5	191·7	8·68	·866	2	yel, blsh.
351	3 2·4	+43 52	8·5,	9·0	117·8	27·50	·866	2	Lewis 43° 40′.
363	,, 11·3	78 9	8·5,	8·7	312·7	26·22	·866	2	
364	,, 7·1	38 46	8·5,	8·5	310·8	11·45	·863	2	
366 *rej.*	,, 9·1	22 35	7·0,	9·7	39·9	46·75	·885	2	AB. ylsh, prpsh; A and D>.
411 *rej.*	,, 26·4	− 7 30	Fr.	7·0,	9·0	88·1	19·35	·885	2	wh, bl (not in Lewis).
414	,, 28·7	+19 28	8·0,	8·0	183·5	7·66	·885	2	=0Σ 58; D<s.
416 *rej.*	,, 29·2	19 29	8·5,	9·7	56·9	27·31	·885	2	yel, bl; D<.
417 *rej.*	,, 28·5	− 2 53	8·5,	9·6	179·2	25·68	·892	2	wh, o.
418	,, 33·3	+75 2	8·5,	9·2	62·7	19·81	·892	2	D<.
422	,, 31·6	0 16	6·0,	8·2	251·0	6·42	·898	2	P. III 98; yel, bl.
452	,, 42·8	10 50	5·0,	9·6	57·8	9·02	·898	2	30 Tauri; wh, o.
456 *rej.*	,, 43·2	1 21	8·5,	9·5	121·3	21·79	·898	2	wh, o; A<?; Lewis 1° 11′.
479	,, 55·0	22 55	7·0,	7·9	127·8	7·28	·892	2	AB. wh, prpsh. } Triple.
,,	,, ,,	,, ,,	,,	9·4	240·7	58·88	·892	2	AC. wh, blue. } Triple.
494	4 2·9	22 49	7·7,	7·7	186·8	5·25	·932	2	ylsh wh, ylsh wh.
501	,, 3·6	− 2 57	8·3,	9·5	296·4	29·25	·939	2	
516	,, 9·6	−10 30	6·0,	9·0	150·4	6·36	·932	2	39 Erid.; p. or, bl.
525 *rej.*	,, 13·6	− 2 56	8·2,	9·2	243·4	44·06	·939	2	AB. } Triple.
,,	,, ,,	,, ,,	9·2,	9·7	167·4	7·39	·939	2	BC. } Triple.

PART I.—*continued.*

Σ.	R.A. 1900. h m	Decl. 1900. ° ′	Magnitudes.		Pos. Ang. °	Distance. ″	Epoch.	Nts.	Remarks.
526	4 17·2	+60 2	8·2,	8·7	53·5	5·62	1916·939	2	
[OΣ 81]	,, 18·1	33 44	6·0,	8·8	40·0	4·65	·939	2	56 Pers.; p. yel, prp; A>.
539 *rej.*	,, 18·3	− 5 14	8·2,	10·2	235·4	17·67	·939	2	yel, bl; >10·2 mg.
542	,, 20·5	+46 1	8·3,	10·0	105·5	21·16	·962	2	
569	,, 30·8	9 0	8·2,	8·7	133·3	8·06	·960	2	
603	,, 46·9	49 26	8·0,	8·2	238·7	8·45	·962	2	
617	,, 54·1	62 52	8·5,	8·7	120·6	12·43	·962	2	
654	5 8·1	2 45	4·7,	8·5	63·7	6·97	1917·108	2	ρ Ori.; p. orge, blue.
674	,, 11·6	20 1 Fr.	7·0,	9·5	149·8	9·69	·108	2	wh, bl; P. V 25; A< D>.
680	,, 13·3	20 2	6·0,	9·8	203·3	8·88	·115	2	yel, purp; P. V 37.
[OΣ 106]	,, 16·8	5 18	6·8,	10·2	41·7	9·37	·121	2	v. yel, bl.
697	,, 17·8	15 57	7·2,	8·2	285·1	25·94	·102	2	wh. lilac.
716	,, 23·1	25 4	5·5,	6·6	202·1	5·02	·121	2	118 Tauri; ylsh wh, p. bl.
718	,, 24·7	49 19	7·2,	7·2	77·5	7·75	·138	2	both ylsh wh; A<?
730	,, 26·5	17 0	6·1,	6·4	140·1	9·32	·138	2	p. yel, bl; D>s?
761	,, 33·5	− 2 37 Fr.	7·5,	8·2	201·3	68·04	·108	2	AB. wh, wh. } Triple
,,	,, ,,	,, ,,	8·2,	8·5	267·0	8·50	·108	2	BC. wh, wh. }
766	,, 34·6	+15 19	6·8,	8·0	275·8	10·16	·138	2	wh, bl.
772	,, 36·1	21 32	8·0,	9·0	243·5	29·96	·102	2	
798	,, 43·3	− 8 25	7·2,	9·2	181·2	20·68	·102	2	wh, blsh.
844	6 2·6	+14 0	8·2,	8·8	8·5	23·68	·138	2	
845	,, 3·9	48 44	5·2,	6·4	354·8	7·86	·138	2	41 Aur.; wh, purp.
926	,, 26·3	5 51	7·3,	8·7	286·8	10·58	·165	2	ylsh wh, lilac.
958	,, 39·8	55 50	6·3,	6·3	258·8	5·12	·165	2	both p. yel.
968	,, 44·9	52 48	8·0,	9·0	287·5	20·69	·165	2	
976	,, 45·5	18 48	8·0,	8·8	117·7	35·84	·165	2	
982	,, 49·0	13 19	5·0,	7·7	158·5	6·98	·165	2	38 Gem.; p. yel, purp; A>D<s?
1035	7 6·1	22 27	7·4,	7·4	41·7	8·54	·198	2	both ylsh.
1066	,, 14·1	22 10	3·2,	8·2	211·5	6·69	·194	3	δ Gem.; p. yel, purp.
1083	,, 19·7	20 42	6·8,	7·8	43·0	6·36	·198	2	p. yel, p. bl.
1110	,, 28·2	32 7 Fr.	2·0,	3·0	217·8	5·47	·182	3	α Gem. (*Castor*); wh, gnsh wh.
1140	,, 42·6	18 35	6·8,	8·5	274·0	6·12	·198	2	yel, v. blue.
1223	8 20·7	27 16	6·3,	6·3	218·0	4·87	·206	2	ϕ^2 Canc.; both ylsh wh.
1224	,, 20·7	24 52	7·1,	7·6	43·7	5·93	·206	2	υ^1 Canc.; crmy wh, flushed.
1248	,, 33·6	62 23	8·3,	8·8	210·1	18·56	·206	2	
1258	,, 36·3	49 13	7·2,	7·4	330·6	9·69	·206	2	wh, wh.

PART I.—*continued.*

Σ.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
1308	9 0·0	− 3 35	7·9, 8·9	84·9	10·43	1917·219	2	wh, wh.
1309	,, 1·4	+ 3 13	8·0, 8·3	273·0	11·37	·219	2	
1332	,, 11·6	24 4	7·2, 7·5	21·0	5·70	·219	2	wh, wh; A<s.
1340	,, 15·7	49 57	6·5, 8·3	319·3	5·98	·219	2	39 Lyn.; wh, bl.
1341	,, 15·7	51 2	8·5, 8·5	267·5	20·61	·219	2	
1346	,, 18·6	54 27	7·0, 8·0	313·6	5·63	·228	2	21 U. Maj.; wh, p. bl.
1424	10 14·4	20 21 Fr.	2·5, 4·0	118·5	3·90	·249	3	γ Leonis; gold. yel, grnsh yel.
1456	,, 33·1	1 46	7·8, 9·5	46·4	13·57	·243	2	wh, o.
1463	,, 36·9	47 12	8·5, 9·0	258·7	7·60	·243	2	
1466	,, 38·1	5 16	6·1, 7·2	240·4	6·90	·243	2	35 Sext.; yel, purp.
1481	,, 46·8	− 6 39	8·0, 8·5	343·2	30·03	·243	2	
1487	,, 50·2	+25 17	5·0, 7·0	108·3	6·38	·249	3	54 Leonis; wh, lilac; A<s.
1512	11 3·2	63 3	8·0, 8·3	50·9	9·55	·276	2	
1523	,, 12·9	32 6 Fr.	4·5, 5·0	110·6	3·06	·264	3	ξ U. Maj.; both ylsh wh.
1548	,, 27·0	− 2 58	7·7, 8·7	126·7	10·48	·265	2	ylsh, blsh.
1553	,, 31·1	+56 41	7·3, 7·8	169·6	5·84	·262	2	wh, wh.
1580	,, 50·3	4 6	8·0, 9·0	261·4	8·66	·286	2	
1582	,, 50·9	22 32	7·7, 9·2	76·3	12·08	·286	2	wh, o.
1600	12 0·5	+52 30	7·0, 8·0	93·7	7·64	·291	2	ylsh wh, blsh wh.
1618	,, 10·0	10 33	8·5, 8·5	245·2	26·02	·297	2	
1619	,, 10·0	− 6 42	7·5, 7·8	277·5	6·98	·297	2	wh, wh; A>.
1632	,, 15·3	+38 27	6·5, 9·7	192·2	10·26	·303	2	p. yel, prpsh.
1633	,, 15·6	27 36	7·1, 7·2	243·9	8·71	·303	2	wh, wh.
1656	,, 28·7	39 10	8·5, 8·5	25·0	26·80	·303	2	
1669	,, 36·1	−12 28	6·0, 6·1	304·6	5·75	·303	2	both ylsh wh.
1681	,, 44·5	+ 4 22	8·5, 8·5	194·5	8·45	·303	2	
1723	13 3·6	39 16	8·0, 9·3	8·8	6·63	·320	2	
1747	,, 23·5	48 16	8·2, 9·5	346·6	14·99	·320	2	
1760	,, 29·8	26 47	8·0, 8·0	64·6	8·53	·320	2	
1776	,, 37·7	46 43	8·0, 8·0	198·9	7·30	·325	2	
1789	,, 49·6	33 19	8·0, 8·2	325·8	6·52	·325	2	
1795	,, 55·3	53 35	6·5, 9·5	3·1	7·61	·325	2	P. XIII 277; wh, bl.
1807	14 6·1	− 2 51	8·0, 8·4	28·8	7·06	·349	2	
1814	,, 7·4	+50 43	8·5, 9·0	256·2	11·27	·349	2	
1873	,, 40·0	8 6	7·8, 8·3	92·9	6·75	·349	2	ylsh, blsh.
1877	,, 40·6	27 30 Fr.	3·0, 5·5	327·8	2·83	·350	3	ε Boötis; p. yel, vivid bl.

PART I.—*continued.*

Σ.	R.A. 1900. h m	Decl. 1900. ° ′	Magnitudes,		Pos. Ang. °	Distance. ″	Epoch.	Nts.	Remarks.
1886	14 46·2	10 7	7·2,	9·2	227·1	7·69	1917·351	2	ylsh wh, o.
1912	15 4·1	5 35	8·5,	9·3	153·9	6·74	·367	2	A>?
1934	,, 13·9	44 6	8·5,	8·5	26·3	7·06	·395	2	A>D<.
1935	,, 16·1	31 4	8·5,	8·7	288·9	8·45	·395	2	
1947	,, 24·5	38 42	8·3,	8·7	27·6	6·88	·395	2	
1979	,, 46·3	22 47	8·5,	9·1	246·2	9·57	·395	2	
1990	,, 54·6	22 5	8·0,	8·5	59·7	56·24	·508	2	AB. } Triple.
,,	,, ,,	,, ,,	8·5,	8·5	208·6	3·81	·508	2	BC. }
2008	16 2·4	− 2 23	8·5,	9·2	59·2	8·85	·417	2	
2016	,, 7·5	+12 9	8·3,	9·7	146·2	7·34	·518	2	Fr. not less than 9·3 mag.
2032	,, 11·0	34 6	5·5,	6·5	217·2	5·00	·417	2	σ Coronæ; p. yel, lilac.
2046	,, 20·0	64 36	8·5,	9·3	223·5	7·87	·417	2	
2048	,, 23·4	− 7 54	6·3,	9·0	301·1	4·77	·518	2	P. XVI 88; p. yel, o.
2068	,, 31·1	+47 28	8·3,	8·3	255·7	5·42	·417	2	
2087	,, 38·4	23 50	8·2,	8·2	290·3	5·83	·417	2	
2182	17 28·3	23 56	8·2,	9·2	356·6	5·26	·445	2	
2192	,, 36·1	29 17	7·5,	9·9	58·8	11·23	·445	2	A>. Ylsh, o.
2230	,, 45·9	7 57	8·2,	8·7	84·3	45·68	·445	2	
2272	18 0·4	+ 2 32	4·5,	6·0	136·5	4·70	·522	3	70 Oph.; yel, dull orge (sure).
2321	,, 24·9	1 7	7·9,	9·5	190·9	6·62	·518	2	wh, o.
2429	,, 54·9	36 17	8·3,	8·9	28·1	5·52	·524	2	
2441	,, 58·9	31 15	7·7,	9·3	276·2	5·33	·524	2	ylsh, blsh.
2443	,, 59·5	14 39	8·2,	8·6	312·3	6·68	·524	2	
2468	19 5·4	8 30	8·2,	9·2	257·1	7·62	·555	2	
2510	,, 18·5	9 19	8·5,	8·5	1·0	8·74	·555	2	
2524	,, 22·5	25 19	8·3,	8·5	95·2	6·63	·564	2	A>.
2534	,, 24·1	36 19	7·8,	8·0	63·6	6·86	·555	2	P. XIX 149; wh, wh.
2543	,, 31·3	5 46	7·0,	9·9	154·3	12·72	·555	2	yel, o.
2617	,, 53·4	75 8	8·5,	9·0	41·4	27·78	·564	2	
2631	20 2·8	20 49	8·0,	9·4	337·0	4·72	·620	2	P. XIX 415; A>s.
2639	,, 5·5	35 10	7·7,	8·7	301·5	5·71	·618	2	ylsh wh, blsh wh.
2686	,, 24·9	9 58	8·3,	9·8	278·9	27·35	·620	2	
2725	,, 41·5	15 32	7·3,	8·0	4·7	5·38	·620	2	ylsh wh, lilac; A<s.
2747	,, 58·4	37 15	8·2,	8·2	263·5	4·70	·620	2	
2798	21 21·4	64 30	7·8,	9·7	145·3	6·44	·639	2	ylsh wh, blsh.
[OΣ 440]	,, 24·7	59 19	6·3,	10·5	182·0	11·87	·639	2	or. yel, bl; Fr. quite 10·0 mag. P. XXI 166.

PART I.—*continued.*

Σ.	R.A. 1900. h m	Decl. 1900. ° ′	Magnitudes.	Pos. Ang. °	Distance. ″	Epoch.	Nts.	Remarks.
2814	21 34·7	35 55	8·3, 9·8	160·6	7·83	1917·639	2	
2873	22 2·0	82 23	6·2, 7·0	71·3	13·82	·642	2	ylsh wh, blsh wh; A>s.
2890	,, 11·2	49 23	8·5, 8·7	10·9	9·38	·666	2	In cl. H VIII 75.
2897	,, 16·8	14 44	8·7, 9·5	100·6	17·49	·666	2	
2898	,, 17·3	10 34	8·3, 9·3	281·8	12·51	·672	2	
2910	,, 23·4	23 1	8·3, 8·8	337·8	5·25	·672	2	A>s.
2964	,, 54·5	− 4 55 Fr.	8·0, 9·0	280·7	8·92	·672	2	Lewis 7·5, 8·0; Σ 7·7, 9·5. Mags. discordant.
2970	,, 57·5	−11 51	8·5, 9·0	38·7	8·28	·672	2	
2895	,, 16·0	+24 26 Fr.	8·5, 9·7	36·9	8·12	·685	2	Too bright for 10·0 mag.
2904	,, 22·0	− 2 17	8·9, 9·4	309·9	7·33	·685	2	
2995	23 11·4	− 2 8	7·7, 8·0	31·0	4·90	·696	2	wh, wh.
3003	,, 13·7	+82 54	8·5, 9·0	268·2	22·78	·688	2	
3011	,, 20·3	76 31	8·5, 8·8	333·7	6·78	·691	2	
3016	,, 23·8	− 7 11 Fr.	8·7, 9·5	316·8	20·33	·696	2	A>s.
3042	,, 46·8	+37 21	7·0, 7·0	86·3	4·88	·691	2	v. wh, v. wh.
3048	,, 52·9	23 47	7·7, 8·8	310·6	9·04	·691	2	P. XXIII 240; yel, bl; A>.

PART II.

Miscellaneous Wide Double Stars.

Syn.	R.A. 1900. h m	Decl. 1900. ° ′	Magnitudes.	Pos. Ang. °	Distance. ″	Epoch.	Nts.	Remarks.
Hh 6	0 14·8	+37 35	7·8, 9·2	19·9	72·98	1916·806	2	p. yel, bl.; A and D<.
,, 22	1 0·7	31 39	6·2, 9·5	292·7	57·30	·818	2	σ^2 Pisc.; p. or, bl; D<.
,, 38	,, 43·0	15 50	8·0, 8·2	326·9	52·90	·818	2	wh, wh.
"Cel. Obj."	,, 44·7	−11 10	5·0, 7·5	249·7	182·48	·835	2	χ Ceti and P. I 182; wh, lilac.
Hh 47	,, 54·3	−23 30	7·2, 7·4	306·9	8·47	·840	2	ylsh wh, ylsh wh.; A<.
,, 50	,, 58·7	− 0 49	6·0, 10·0	193·2	42·91	·846	2	61 Ceti; p. orge, o.
,, 62	2 14·3	− 3 26	5·0,v 8·2	78·2	116·34	·824	2	*Mira* Ceti; fine orge, bl; A>.
h 1123	,, 35·3	+42 22	8·1, 8·1	248·5	20·09	·852	2	wh, wh; in cluster M 34.
S 423	,, 52·8	−25 22	7·8, 8·0	222·4	28·15	·870	2	wh, wh.
Hh 90	3 26·8	+11 5	7·5, 9·5	248·2	18·22	·885	2	yel, bl; = Σ 409 *rej.*
S 437	,, 40·2	23 53	8·0, 8·0	303·6	37·12	·892	2	wh, wh; in *Pleiades.*
Hh 117	4 19·7	17 42	5·0, 10·0	236·3	79·72	1917·072	2	δ^3 Tauri; wh, o; D<.

PART II.—*continued.*

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
"Cel. Obj."	4 40·2	18 53	8·5, 8·7	112·7	33·82	1917·072	2	wh, wh; in cluster H VIII 8.
Hh 136	,, 45·3	6 47	7·5, 9·0	4·2	95·67	·072	2	ylsh wh, blsh.
"Cel. Obj."	,, 58·4	23 38	8·5, 9·0	187·3	19·52	·072	2	wh, wh; in cluster H VII 21.
Hh 146	,, 58·8	51 29	5·0, 9·0	60·7	91·11	·082	2	9 Aur., AC.; ylsh wh, bl.
,, 155	5 9·3	45 54	0·2, 10·0	144·0	141·86	·095	2	*a* Aur. (*Capella*); p. yel, o; D>.
,, 171	,, 19·7	− 2 55	7·7, 8·5	281·8	135·72	·095	2	ylsh, blsh.
"Cel. Obj."	,, 24·1	− 4 46	8·5,[v] 9·5	228·1	45·77	·101	2	S Ori.; orge red, bl.
Hh 206	,, 49·8	+ 7 23	1·0, 10·5	153·3	174·42	·101	2	*a* Ori. (Betelgeuse); orge, o.
,, 216	6 4·4	15 56	7·7, 9·2	108·0	54·38	·158	2	wh, blsh.
,, 217	,, 6·0	19 49	6·0, 8·6	213·0	86·10	·158	2	68 Ori.; wh, blsh.
,, 222	,, 19·8	−16 12	7·0, 8·0	87·0	15·86	·158	2	wh, wh.
,, 235	,, 28·1	+22 17	6·7, 6·8	243·0	53·55	·181	2	wh, wh. AB. } Triple.
,, ,,	,, ,,	,, ,,	,,	9·0	148·7	105·83	·181	2	wh, blue. AC. }
,, 266	7 21·8	21 39	5·3, 9·5	324·0	42·89	·197	2	63 Gem.; ylsh wh, o.
,, 273	,, 34·7	−26 34	4·5, 4·6	318·5	9·55	·189	2	*k* Argûs; both p. yel.
,, 280	,, 42·3	+29 2	6·5, 8·5	359·6	89·56	·197	2	orange, blue.
,, 291	8 8·5	17 58	6·4, 9·3	301·9	63·63	·206	2	P. VIII 14; ylsh. wh, o.
,, 293	,, 20·4	28 14	5·8, 9·0	21·0	130·40	·210	2	ϕ^1 Canc.; p. orge, bl.
S 568	,, 20·8	−23 42	5·5, 8·0	86·3	42·13	·208	2	P. VIII 72–4; orge, lilac.
,, 571	,, 34·7	+19 54	7·0, 7·2	156·2	45·68	·206	2	P. VIII 124; both blsh wh.
Hh 308	,, 46·4	32 51	5·7, 9·0	22·8	80·28	·210	2	σ^1 Canc.; ylsh wh, o.
,, 338	9 30·5	14 49	6·2, 8·5	79·4	40·58	·231	2	7 Leon.; y. wh, blsh.
Hh 341	,, 35·8	+10 21	4·0, 9·2	44·0	85·34	·231	2	14 Leon.; y. wh, o.; A and D<.
,, 344	,, 59·3	−17 37	7·0, 7·5	273·5	21·00	·231	2	ylsh wh, blsh wh.
,, 350	10 18·1	+ 6 12	6·5, 8·7	350·4	57·31	·234	2	ylsh wh, blsh; A< D>s.
"Cel. Obj."	,, 30·8	12 11	8·0, 9·0	337·0	56·70	·241	2	yel, bl.
Hh 363	11 10·7	−15 48	7·9, 8·3	301·3	19·47	·276	2	wh, wh.
,, 364	,, 11·6	− 3 6	4·6, 8·5	289·0	97·60	·265	2	ϕ Leon.; ylsh wh, bl.
,, 373	,, 24·0	−16 46	7·0, 8·0	330·7	28·50	·265	2	wh, wh.
,, 376	,, 27·3	−28 43	5·8, 5·9	211·4	8·76	·268	2	17 Crat.; yel, yel.
h 508	,, 36·9	+40 14	8·3, 9·3	127·9	11·16	·286	2	wh, o.
Hh 379	,, 43·3	− 9 44	6·5, 8·6	68·7	84·86	·286	2	yel, bl; D< ?
,, 384	,, 47·3	+16 0	7·0, 9·0	13·1	39·66	·286	2	ylsh, o; A>D<.

PART II.—*continued.*

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.	Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′		°	″			
Hh 399	12 33·6	− 3 50	8·0, 9·3	112·4	51·90	1917·310	2	wh, o; A<.
,, 401	,, 36·6	+10 58	6·3, 9·4	282·1	84·91	·310	2	27 Virg.; ylsh wh, blsh.
S 642	,, 43·8	14 36	8·4, 9·0	28·5	59·92	·310	2	wh, wh; A>D<.
Hh 412	13 8·1	−18 18	6·5, 7·0	33·9	5·09	·343	2	54 Virg.; ylsh, blsh.
,, 421	,, 27·3	−12 10	7·0, 7·8	80·0	47·91	·336	2	wh, wh.
,, 422	,, 31·2	−25 59	5·7, 7·6	190·6	9·90	·343	2	ylsh wh, bl.
,, 430	,, 49·9	+18 54	2·8, 9·0	112·7	112·63	·327	2	η Boötis; ylsh wh, bl.
,, 435	14 4·6	49 56	5·4, 10·0	274·5	79·68	·336	2	13 Boötis; p. orge, o.
,, 443	,, 20·0	−19 31	6·5, 7·0	294·8	34·65	·343	2	both ylsh.
,, 449	,, 40·2	−25 1	5·5, 7·5	128·5	8·80	·357	2	54 Hyd.; p. yel, prpsh.
,, 452	,, 43·6	−23 50	6·0, 8·5	219·9	60·74	·367	2	p. orge, bl.
,, 460	,, 52·4	+ 0 14	5·7, 9·5	224·3	86·79	·357	2	1 Serp.; p. yel, bl.
,, 472	15 15·4	−15 12	6·5, 8·7	351·4	47·11	·364	2	o^1 Lib.; p. yel, blsh.
,, 471	,, 22·8	− 8 59	7·2, 7·5	133·6	51·94	·367	2	wh, wh.
S 672	,, 26·9	−19 53	6·8, 8·5	280·0	10·91	·508	2	yel, bl.
Hh 486	,, 52·1	+38 14	5·5, 9·5	64·5	97·78	·392	2	λ Cor.; wh, o.
Sh 213	,, 54·4	−19 43	7·5, 7·7	318·8	17·65	·508	2	wh, wh.
Hh 489	,, 57·0	+26 27	8·0, 10·0	110·0	48·13	·370	2	wh, o.
,, 490	,, 57·2	33 37	5·4, 9·0	74·5	84·21	·370	2	ρ Cor.; ylsh wh, blsh.
,, 502	16 12·7	29 23	5·7, 9·3	21·6	86·22	·417	2	AC. υ Cor.; wh, o, o. } Quintuple.
,, ,,	,, ,,	,, ,,	,, 9·0	51·3	122·92	·417	2	AD. } Quintuple.
,, 512	,, 19·6	−23 13	5·2, 5·9	354·2	3·52	·417	2	ρ Oph.; p. yel, p bl.
Del. Obj."	,, 51·2	−19 24	6·3, 8·0	229·3	5·96	·417	2	P. XVI 236; wh, prpsh.
h 589	17 4·7	−24 49	7·5, 7·8	303·6	10·07	·417	2	wh, wh.
Del. Obj."	,, 9·2	−26 27	5·5, 5·5	184·2	4·21	·417	2	36 Oph.; yel, yel. A retrog.
Hh 603	19 2·5	+35 42	8·0, 9·5	56·3	56·05	·524	2	wh, blsh.
,, 644	,, 45·7	17 40	8·0, 10·0	83·6	24·49	·564	2	AB. wh, o. } Triple.
,, ,,	,, ,,	,, ,,	,, 8·6	256·4	68·34	·564	2	AC. wh, blsh } Triple.
,, 673	20 6·4	−12 22	8·0, 9·0	264·7	85·02	·619	2	ylsh wh, bl.
,, 730	21 21·7	+36 14	6·0, 9·5	98·0	53·31	·624	2	69 Cygni; ylsh wh, o.
,, 734	,, 31·5	−19 54	5·0, 9·0	47·0	67·98	·627	2	ϵ Capr.; p. yel, bl.
,, 741	,, 37·6	+40 21	6·0, 9·0	231·4	62·31	·624	2	76 Cygni; wh, bl.
,, 745	,, 39·3	37 50	5·5, 9·0	319·3	134·32 AB	·621	2	79 Cygni; wh, blsh. } Triple.
,, ,,	,, ,,	,, ,,	,, 7·0	59·2	150·92 AC	·621	2	wh. yel. } Triple.
,, 764	22 24·8	57 55	8·5, 9·8	9·7	25·19	·664	2	ylsh, o; *p.* δ Cephei.

PART II.—*continued.*

Syn.	R.A. 1900.	Decl. 1900.	Magnitudes.		Pos. Ang.	Distance.	Epoch.	Nts.	Remarks.
	h m	° ′			°	″			
Hh 771	22 34·8	38 32	5·0,	9·5	49·5	61·67	1917·664	2	10 Lacertæ; grnsh wh, o.
,, 772	,, 37·0	39 43	5·5,	9·5	15·9	69·42	·664	2	12 Lacertæ; wh, o.
,, 790	23 5·4	58 47	5·6,	8·3	162·9	167·09	·675	2	2 Cassiop.; y. wh, p. bl.
,, 797	,, 20·4	61 44	5·2,	8·5	224·8	97·55 AB	·675	2	4 Cassiop.; orge, bl. } Triple.
,, ,,	,, ,,	,, ,,	,,	10·0	257·8	215·03 AC	·675	2	orge, o. }
,, 799	,, 22·9	5 30	8·0,	9·0	2·6	88·48	·675	2	p. yel, bl; A>.

Note.

Abbreviations used.—<, increasing; >, decreasing; A (where combined with the foregoing symbols), angle; D, distance. Where A is otherwise used it refers to the brighter component of a double star. Cols., colours; orge., orange; yel., yellow; ylsh., yellowish; grn., green; grnsh., greenish; bl., blue; blsh., bluish; prp., purple; prpsh., purplish; wh., white; v. (where prefixed to a colour), very; p. pale; o, no colour seen. Where v. is affixed to a magnitude determination it means *variable*. A small "s" following the symbol < or > means that the inferred change is *slight* or *slow*.

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Erratum.

In Mr. Whitmell's paper on the Solar Eclipse of 1917 December 13, vol. lxxvi. p. 410, 8th line from foot:—

for a narrow band, 45 miles wide, *read* a band, 115 miles wide.